Mathematics Standards for High School

The high school standards specify the mathematics that all students should study in order to be college and career ready. Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics is indicated by (+), as in this example:

(+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers).

All standards without a (+) symbol should be in the common mathematics curriculum for all college and career ready students. Standards with a (+) symbol may also appear in courses intended for all students.

The high school standards are listed in conceptual categories:

Number and Quantity
Algebra
Functions
Modeling
Geometry
Statistics and Probability

Conceptual categories portray a coherent view of high school mathematics; a student’s work with functions, for example, crosses a number of traditional course boundaries, potentially up through and including calculus.

Modeling is best interpreted not as a collection of isolated topics but in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★). The star symbol sometimes appears on the heading for a group of standards; in that case, it should be understood to apply to all standards in that group.
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<tr>
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Numbers and Number Systems. During the years from kindergarten to eighth grade, students must repeatedly extend their conception of number. At first, "number" means "counting number": 1, 2, 3... Soon after that, 0 is used to represent "none" and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers.

With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in two important ways: They have the commutative, associative, and distributive properties and their new meanings are consistent with their previous meanings.

Extending the properties of whole-number exponents leads to new and productive notation. For example, properties of whole-number exponents suggest that \((5^{1/3})^3\) should be \(5^{(1/3)\times3} = 5^1 = 5\) and that \(5^{1/3}\) should be the cube root of 5.

Calculators, spreadsheets, and computer algebra systems can provide ways for students to become better acquainted with these new number systems and their notation. They can be used to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

Quantities. In real world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages. They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process is sometimes called quantification. Quantification is important for science, as when surface area suddenly “stands out” as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.
Number and Quantity Overview

The Real Number System

- Extend the properties of exponents to rational exponents
- Use properties of rational and irrational numbers.

Quantities

- Reason quantitatively and use units to solve problems
- (IA) Understand and apply the mathematics of voting.
- (IA) Understand and apply some basic mathematics of information processing and the Internet.

The Complex Number System

- Perform arithmetic operations with complex numbers
- Represent complex numbers and their operations on the complex plane
- Use complex numbers in polynomial identities and equations

Vector and Matrix Quantities

- Represent and model with vector quantities.
- Perform operations on vectors.
- Perform operations on matrices and use matrices in applications.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
Extend the properties of exponents to rational exponents. (N-RN.A)

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{1(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5. (N-RN.A.1) (DOK 1,2)
   a. Example: Solution (DOK 3)

A biology student is studying bacterial growth. She was surprised to find that the population of the bacteria doubled every hour.

a. Complete the following table and plot the data.

<table>
<thead>
<tr>
<th>Hours into study</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population (thousands)</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Write an equation for $P$, the population of the bacteria, as a function of time, $t$, and verify that it produces correct populations for $t = 1, 2, 3,$ and 4.

c. The student conducting the study wants to create a table with more entries; specifically, she wants to fill in the population at each half hour. However, she forgot to make these measurements so she wants to estimate the values.

Instead she notes that the population increases by the same factor each hour, and reasons that this property should hold over each half-hour interval as well. Fill in the part of the table below that you've already computed, and decide what constant factor she should multiply the population by each half hour in order to produce consistent results. Use this multiplier to complete the table:

<table>
<thead>
<tr>
<th>Hours into study</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population (thousands)</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
d. What if the student wanted to estimate the population every 20 minutes instead of every 30 minutes. What multiplier would be necessary to be consistent with the population doubling every hour? Use this multiplier to complete the following table:

<table>
<thead>
<tr>
<th>Hours into study</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

 e. Use the population context to explain why it makes sense that we define $2^{\frac{1}{2}}$ to be $\sqrt{2}$ and $2^{\frac{3}{2}}$ as $\sqrt{2}$.

 f. Another student working on the bacteria population problem makes the following claim:

   *If the population doubles in 1 hour, then half that growth occurs in the first half-hour and the other half occurs in the second half-hour. So for example, we can find the population at $t = \frac{1}{2}$ by finding the average of the populations at $t = 0$ and $t = 1$.\'*

Comment on this idea. How does it compare to the multipliers generated in the previous problems? For what kind of function would this reasoning work?

b. Example: Solution (DOK 3)

Three students disagree about what value to assign to the expression $0^0$. In each case, critically analyze the student’s argument.

a. Juan suggests that $0^0 = 1$:

   *I know that $2^0 = 1$ and $1^0 = 1$ and $x^0 = 1$ for any non-zero real number x. So $0^0$ should also be 1.*

b. Briana thinks that $0^0 = 0$:

   *I know that $0^1 = 0$ and $0^2 = 0$ and $0^x = 0$ for any non-zero real number x. So $0^0$ should be 0.*

c. Kristin says that $0^0 = -1$:

   *If I try some negative numbers, I find $(\frac{1}{3})^{-\frac{1}{3}}$ is about -1.44, $(\frac{1}{5})^{-\frac{1}{5}}$ is about -1.38, and $(\frac{1}{1000})^{-\frac{1}{1000}} = -1.007$. As the base and exponent both get closer to 0, it looks like the values are getting closer and closer to -1. So $0^0$ should be -1.*

c. Example: Solution (DOK 3)
Henry explains why $4^{3/2} = 8$:

*I know that $4^3$ is 64 and the square root of 64 is 8.*

Here is Henrietta’s explanation for why $4^{3/2} = 8$:

*I know that $\sqrt[3]{4} = 2$ and the cube of 2 is 8.*


b. Calculate $4^{6/2}$ and $2^{7/3}$ using Henry's or Henrietta's strategy.

c. Use both Henry and Henrietta's reasoning to express $x^{m/n}$ using radicals (here $m$ and $n$ are positive integers and we assume $x > 0$).

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents. (N-RN.A.2) (DOK 1)

a. Example: Solution (DOK 3)

In each of the following problems, a number is given. If possible, determine whether the given number is rational or irrational. In some cases, it may be impossible to determine whether the given number is rational or irrational. Justify your answers.

a. $4 + \sqrt{7}$

b. $\frac{\sqrt{45}}{\sqrt{5}}$

c. $\frac{6}{\pi}$

d. $\sqrt{2} + \sqrt{3}$

e. $\frac{2 + \sqrt{7}}{2a + \sqrt{7}a^2}$, where $a$ is a positive integer

f. $x + y$, where $x$ and $y$ are irrational numbers

b. Example: Solution (DOK 3)

Alicia and Zara are scientists working together. Alicia uses a calculator to evaluate $3^{1.4}$ and gets an answer of 6.473. Zara thinks for a moment, makes some calculations on paper, and says "That cannot be right, because $3^{1.4}$ must be less than 6."

Find some hand calculations which show that, as Zara says, $3^{1.4}$ must be less than 6.

c. Example: Solution (DOK 2)
Below is a picture of the (elliptical) orbit of a planet around the sun:

The sun is at point \( A \), point \( P \) is where the planet is closest to the sun during its orbit, and point \( Q \) is where the planet is farthest from the sun during its orbit. Kepler made the following amazing discovery: if \( a \) is the average of the closest and farthest distances of the planet from the sun and \( t \) is the time it takes the planet to make one full orbit around the sun then the quotient \( \frac{t^2}{a^3} \) does not depend on the planet. In what follows, we measure \( t \) in years and \( a \) in astronomical units where one astronomical unit is the average of the closest and farthest distances from the earth to the sun. These units are helpful since we have \( t = 1 \) and \( a = 1 \) for the earth, allowing us to find the quotient \( \frac{t^2}{a^3} = 1 \).

a. Find an equation for \( t \) in terms of \( a \) and an equation for \( a \) in terms of \( t \).

b. The orbit of Mars takes about 1.88 years. What is its average distance from the sun?

c. The average distance of Neptune is 30.06 astronomical units. About how long does each orbit of Neptune take?

d. What is the farthest a planet could be from the sun during its orbit if each orbit takes 5 years?

d. Example: Write the value of \( x \) such that \( \frac{4}{5} \cdot \frac{3}{x} = \frac{5}{\sqrt{37}} \) is true.
Use properties of rational and irrational numbers. (N-RN.B)

**Example:** Ashley claims that when you multiply two different square roots together, the product is always rational.

For example, \( \sqrt{2} \cdot \sqrt{18} = \sqrt{36} = 6 \) and \( \sqrt{3} \cdot \sqrt{27} = \sqrt{81} = 9 \)

She also claims that when you multiply two different cube roots together, the product is always irrational.

For example, \( \sqrt[3]{2} \cdot \sqrt[3]{18} = \sqrt[3]{36} \approx 3.3019 \) and \( \sqrt[3]{3} \cdot \sqrt[3]{27} = \sqrt[3]{81} \approx 4.3267 \)

Which statement correctly classifies Ashley’s claims and provides appropriate reasoning?

a. Ashley is correct because her examples support both claims.

b. Ashley is correct about the product of square roots always being rational, but the product of cube roots can sometimes be rational.

c. Ashley is incorrect about the product of square roots always being rational, but she is correct that the product of cube roots is always irrational.

d. Ashley is incorrect because sometimes the product of square roots can be irrational and sometimes the product of cube roots can be rational.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. (N-RN.B.3) (DOK 1,2)

a. Example: Solution (DOK 2)

   Experiment with sums and products of two numbers from the following list to answer the questions that follow:

   \[ 5, \frac{1}{2}, 0, \sqrt{2}, -\sqrt{2}, \frac{1}{\sqrt{2}}, \pi. \]

   Based on the above information, conjecture which of the statements is ALWAYS true, which is SOMETIMES true, and which is NEVER true?

   a. The sum of a rational number and a rational number is rational.
   b. The sum of a rational number and an irrational number is irrational.
   c. The sum of an irrational number and an irrational number is irrational.
   d. The product of a rational number and a rational number is rational.
   e. The product of a rational number and an irrational number is irrational.
   f. The product of an irrational number and an irrational number is irrational.

b. Example: Solution (DOK 3)
a. Explain why the sum and product of two rational numbers is always a rational number.

b. Kaylee says

\[ I\ know\ that\ \pi\ is\ an\ irrational\ number\ so\ its\ decimal\ never\ repeats.\ I\ also\ know\ that\ \frac{1}{7}\ is\ a\ rational\ number\ so\ its\ decimal\ repeats.\ But\ I\ don't\ know\ how\ to\ add\ or\ multiply\ these\ decimals\ so\ I\ am\ not\ sure\ if\ \pi + \frac{1}{7}\ and\ \pi \times \frac{1}{7}\ are\ rational\ or\ irrational. \]

Using part (a), help Kaylee decide whether or not \( \pi + \frac{1}{7} \) and \( \pi \times \frac{1}{7} \) are rational or irrational.

c. Example: Circle \textbf{two} numbers whose product is \textbf{irrational}.

<table>
<thead>
<tr>
<th>Numbers</th>
<th>(-5)</th>
<th>(\frac{1}{3})</th>
<th>(\frac{2}{3})</th>
<th>(3\sqrt{2})</th>
<th>(\sqrt{8})</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Item</th>
<th>Claim</th>
<th>Domain</th>
<th>Target</th>
<th>DOK</th>
<th>CONTENT</th>
<th>MP</th>
<th>Key</th>
</tr>
</thead>
<tbody>
<tr>
<td>#4</td>
<td>1</td>
<td>N-RN</td>
<td>B</td>
<td>2</td>
<td>HSN.RN.B.3</td>
<td>N/A</td>
<td>Numbers</td>
</tr>
</tbody>
</table>
**Quantities**

Reason quantitatively and use units to solve problems. (N-Q.A)

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. (N-Q.A.1) (DOK 1,2)
   
   a. Example: Solution (DOK 3)

According to an article in Runners' World magazine:

> On average the human body is more than 50 percent water [by weight]. Runners and other endurance athletes average around 60 percent. This equals about 120 soda cans' worth of water in a 160-pound runner!

Investigate their calculation. Approximately how many soda cans' worth of water are in the body of a 160-pound runner? What unprovided information do you need to answer this question?

b. Example: Solution (DOK 2)

As Felicia gets on the freeway to drive to her cousin's house, she notices that she is a little low on gas. There is a gas station at the exit she normally takes, and she wonders if she will have to get gas before then. She normally sets her cruise control at the speed limit of 70mph and the freeway portion of the drive takes about an hour and 15 minutes. Her car gets about 30 miles per gallon on the freeway, and gas costs $3.50 per gallon.

   a. Describe an estimate that Felicia might do in her head while driving to decide how many gallons of gas she needs to make it to the gas station at the other end.

   b. Assuming she makes it, how much does Felicia spend per mile on the freeway?

   c. Example: Solution (DOK 3)
A liquid weed-killer comes in four different bottles, all with the same active ingredient. The accompanying table gives information about the concentration of active ingredient in the bottles, the size of the bottles, and the price of the bottles. Each bottle's contents is made up of active ingredient and water.

<table>
<thead>
<tr>
<th>Concentration</th>
<th>Amount in Bottle</th>
<th>Price of Bottle</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.04%</td>
<td>64 fl oz</td>
</tr>
<tr>
<td>B</td>
<td>18.00%</td>
<td>32 fl oz</td>
</tr>
<tr>
<td>C</td>
<td>41.00%</td>
<td>32 fl oz</td>
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<tr>
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b. The size of your lawn requires a total of 14 fl oz of active ingredient. Approximately how much would you need to spend if you bought only the A bottles? Only the B bottles? Only the C bottles? Only the D bottles?

Supposing you can only buy one type of bottle, which type should you buy so that the total cost to you is the least for this particular application of weed killer?

d. Example: Solution (DOK 2)

A team of farm-workers was assigned the task of harvesting two fields, one twice the size of the other. They worked for the first half of the day on the larger field. Then the team split into two groups of equal number. The first group continued working in the larger field and finished it by evening. The second group harvested the smaller field, but did not finish by evening. The next day one farm-worker finished the smaller field in a single day's work. How many farm-workers were on the team?

e. Example: Solution (DOK 3)
You are considering driving an ice cream van during the summer vacation. Your friend, who “knows everything” tells you that “It's easy money.” You make a few inquiries and find that the van costs $600 per week to rent. Each ice cream cone costs 50 cents to make and sells for $1.50.

For each of the questions below, show all work and include an explanation of your method of solution.

a. How many ice cream cones would you have to sell each week just to cover the cost of renting the van?

b. In order to sell the ice cream cones, you have a choice of driving the van through neighborhoods or parking the van in a public area. Typical selling data is that one can sell an average of 35 ice cream cones per hour at each of your planned stops if driving through neighborhoods, while you can sell an average of 30 ice cream cones per hour if one parks the van in a public area.

   i. If you choose to drive the van, you will have to consider the time spent driving the van, which will depend on the average speed from stop to stop on your route, as well as the cost of gasoline, which will depend on the number of miles per gallon the van gets. Make reasonable estimates for these and any other costs you feel would be relevant. If you drive an average of 180 miles per week, how many ice cream cones would you have to sell just to cover the cost of driving the van for a week (not including rental costs)?

   ii. If you choose to park the van, you will have to pay a one-time seasonal permit fee and weekly space rental. If the seasonal permit costs $90.00 and space rental ranges from $140 to $150 per week, how many ice cream cones would you have to sell just to cover the cost of parking the van for a week (again, not including rental costs)? Identify any assumptions you make.

c. How many hours a week will you have to work in order to make this “easy money”? After how many hours would the amounts you earned under each of the two options be the same? How much money might you be able to make if you were willing to work really hard? Identify and take into account any additional expenses for the additional hours. Explain your reasoning clearly.

f. Example: Solution (DOK 2)
The price of copper fluctuates. Between 2002 and 2011, there were times when its price was lower than $1.00 per pound and other times when its price was higher than $4.00 per pound. Copper pennies minted between 1962 and 1982 are 95% copper and 5% zinc by weight, and each penny weighs 3.11 grams. At what price per pound of copper does such a penny contain exactly one cent worth of copper? (There are 454 grams in one pound.)

g. Example: **Solution** (DOK 2)

A fuel oil dealer buys 20,000 gallons of heating oil at $2.65 per gallon and another 14,000 gallons at $3.00 per gallon. (The oil is the same grade and quality, but the price varies due to the market.) He has a contract to sell up to 35,000 gallons of oil next month at $3.25 per gallon, but wants to use as much cash as possible immediately for future investments. To raise cash, he can sell some of his oil to another distributor, who will pay $2.75 per gallon now. How much investment money can the dealer raise now by selling oil and still be able to break even after selling the remainder next month?

h. Example: **Solution** (DOK 2)

Sadie has a cousin Nanette in Germany. Both families recently bought new cars and the two girls are comparing how fuel efficient the two cars are. Sadie tells Nanette that her family’s car is getting 42 miles per gallon. Nanette has no idea how that compares to her family’s car because in Germany mileage is measured differently. She tells Sadie that her family’s car uses 6 liters per 100 km. Which car is more fuel efficient?

i. Example: **Solution** (DOK 3)

One online source suggests that exploiting solar energy makes sense in an area that receives $9 \text{ kWh/m}^2$ of solar energy per day and does not make sense in an area that receives only $2 \text{ kWh/m}^2$ of solar energy per day. Does it make sense to exploit solar energy in Santa Rosa?

j. Example: Given the formula $K = \frac{1}{2}mv^2$, where

- $K$ represents kinetic energy,
- $m$ represents mass and has units of kilograms (kg), and
- $v$ represents velocity and has units of meters per second (m/s)

Select an appropriate measurement unit for kinetic energy.

a. $\frac{kg \cdot m}{s^2}$
b. \( \frac{kg \; m^2}{s} \)

c. \( \frac{kg \; m^2}{s^2} \)

d. \( \frac{kg^2 \; m^2}{s^2} \)

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<td>N-Q</td>
<td>C</td>
<td>1</td>
<td>HSN.Q.A.1</td>
<td>N/A</td>
<td>C</td>
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2. Define appropriate quantities for the purpose of descriptive modeling. \( \text{(N-Q.A.2) (DOK 1,2)} \)
   a. Example: Solution (DOK 3)

A liquid weed-killer comes in four different bottles, all with the same active ingredient. The accompanying table gives information about the concentration of active ingredient in the bottles, the size of the bottles, and the price of the bottles. Each bottle's contents is made up of active ingredient and water.

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c. Example: Solution (DOK 3)

A small company wants to give raises to their 5 employees. They have $10,000 available to distribute. Imagine you are in charge of deciding how the raises should be determined.

a. What are some variables you should consider?

b. Describe mathematically different methods to distribute the raises.

c. What information do you need to compute the raises for each employee?

d. Make up the information you need to compute specific raises for 2 different methods and apply them to the situation. Compute the specific dollar mount each employee receives as a raise.

e. Choose one of you methods that you think is most fair and construct an argument that supports your decision.

d. Example: There is a traffic jam on a highway. From an aerial view, a reporter is trying to estimate the number of vehicles stuck in the traffic jam.

Select all information that will help the reporter make a reasonable estimate of the number of vehicles in the traffic jam.

a. The cause of the traffic jam
b. The average length of a vehicle
c. The number of lanes on the highway
d. The average distance between vehicles
e. The average number of people in each vehicle
f. The distance from the beginning to the end of the traffic jam

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3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. (N-Q.A.3) (DOK 1,2)

a. Example: Solution (DOK 3)
Quincy is a tour guide at a museum of science and history. During a tour of the museum, he tells some visitors about a fossilized dinosaur bone that is on display in the museum. He says, “Twenty years ago, a group of paleontologists donated this dinosaur bone to our museum. At the time, they told us that they had estimated the age of the bone to be approximately 90 million years. So now, the bone is about 90 million and 20 years old.” Evaluate the validity of Quincy’s statement.

b. Example: Solution (DOK 3)

Julio went to Germany to watch an international soccer tournament. He first watched Argentina play Germany in Berlin, Germany. The next day Julio went to Frankfurt, Germany to watch Brazil play France.

To get from Berlin to Frankfurt for this second game, Julio took a bus from Berlin to Erfurt (303 km); then he rented a car and drove from Erfurt to Frankfurt (254 km). Julio drove on German highways, called autobahns, which have no general speed limit for passenger vehicles; however, buses have an enforced speed limit of 80 km/hr.

a. If the bus drove 80 km/hr from Berlin to Erfurt and Julio drove 130 km/hr from Erfurt to Frankfurt, what was the total amount of time it took Julio to travel from Berlin to Frankfurt (not counting the transfer time between the bus and the car)? Give your answer to a reasonable level of accuracy.

b. What fraction of the time during the trip did Julio spend on the bus? In the car?

c. What was Julio’s average speed for the entire trip?

d. If Julio had rented a car and driven from Berlin to Erfurt at 130 km/hr and then taken a bus from Erfurt to Frankfurt at 80 km/hr, would he have arrived sooner? Explain your answer.

e. How long would Julio’s trip have taken if he had ridden the bus the entire way?

c. Example: Solution (DOK 3)
As Felicia gets on the freeway to drive to her cousin’s house, she notices that she is a little low on gas. There is a gas station at the exit she normally takes, and she wonders if she will have to get gas before then. She normally sets her cruise control at the speed limit of 70 mph and the freeway portion of the drive takes about an hour and 15 minutes. Her car gets about 30 miles per gallon on the freeway, and gas costs $3.50 per gallon.

a. Describe an estimate that Felicia might do in her head while driving to decide how many gallons of gas she needs to make it to the gas station at the other end.

b. Assuming she makes it, how much does Felicia spend per mile on the freeway?

d. Example: Solution (DOK 3)

A liquid weed-killer comes in four different bottles, all with the same active ingredient. The accompanying table gives information about the concentration of active ingredient in the bottles, the size of the bottles, and the price of the bottles. Each bottle’s contents is made up of active ingredient and water.

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e. Example: Solution (DOK 3)
The half-life of Carbon 14, that is, the time required for half of the Carbon 14 in a sample to decay, is variable: not every Carbon 14 specimen has exactly the same half life. The half-life for Carbon 14 has a distribution that is approximately normal with a standard deviation of 40 years. This explains why the Wikipedia article on Carbon 14 lists the half-life of Carbon 14 as $5730 \pm 40$ years. Other resources report this half-life as the absolute amounts of 5730 years, or sometimes simply 5700 years.

a. Explain the meaning of these three quantities ($5730 \pm 40$, 5730, and 5700) focusing on how they differ.

b. Can all three of the reported half-lives for Carbon 14 be correct? Explain.

c. What are some of the benefits and drawbacks for each of the three ways of describing the half-life of Carbon 14?

f. Example: Solution (DOK 3)
Carbon 14 is a form of carbon which decays exponentially over time. The amount of Carbon 14 contained in a preserved plant is modeled by the equation

$$f(t) = 10 \left( \frac{1}{2} \right)^{ct}.$$  

Time in this equation is measured in years from the moment when the plant dies ($t = 0$) and the amount of Carbon 14 remaining in the preserved plant is measured in micrograms (a microgram is one millionth of a gram). The number $c$ in the exponential measures the exponential rate of decay of Carbon 14.

a. How many micrograms of Carbon 14 are in the plant at the time it died?

b. The best known estimate for the half-life of Carbon 14, that is the amount of time it takes for half of the Carbon 14 to decay, is $5730 \pm 40$ years. Use this information to calculate the range of possible values for the constant $c$ in the equation for $f$.

c. Use your answer from part (b) to find the range of years when there is one microgram remaining in the preserved plant.

g. Example: Solution (DOK 3)
The label on a 16.9 ounce bottle of sports drink indicates that one serving of 8 ounces contains 50 calories.

a. Based on this information, about how many calories are in the full bottle?

b. The label also says that the full bottle contains 120 calories. Does this agree with your estimate from part (a)? How can you explain the discrepancy (if there is a discrepancy)?

c. The label on a 20 ounce bottle of the same sports drink says the bottle contains 130 calories. Is this consistent with the information on the 16.9 ounce bottle?

(IA) Understand and apply the mathematics of voting.

IA.3. Understand, analyze, apply, and evaluate some common voting and analysis methods in addition to majority and plurality, such as runoff, approval, the so-called instant-runoff voting (IRV) method, the Borda method and the Condorcet method. (N-Q.B.IA.3) (DOK 1,2,3)

(IA) Understand and apply some basic mathematics of information processing and the Internet.

IA.4. (+) Describe the role of mathematics in information processing, particularly with respect to the Internet. (N-Q.C.IA.4) (DOK 1)

IA.5. (+) Understand and apply elementary set theory and logic as used in simple Internet searches. (N-Q.C.IA.5) (DOK 1,2)

IA.6. (+) Understand and apply basic number theory, including modular arithmetic, for example, as used in keeping information secure through public-key cryptography. (N-Q.C.IA.6) (DOK 1,2)
Perform arithmetic operations with complex numbers. (N-CN.A)

1. Know there is a complex number \( i \) such that \( i^2 = -1 \), and every complex number has the form \( a + bi \) with \( a \) and \( b \) real. **(N-CN.A.1) (DOK 1)**
   a. Example: **Solution** (DOK 3)
      
      For this task, the letter \( i \) denotes the imaginary unit, that is, \( i = \sqrt{-1} \).

      a. For each integer \( k \) from 0 to 8, write \( i^k \) in the form \( a + bi \).

      b. Describe the pattern you observe, and algebraically prove your observation. In particular, simplify \( i^{195} \).

      c. Write each of the following expression in the form \( a + bi \):

      - \( i^2 + i + 1 \)
      - \( i^3 + i^2 + i + 1 \)
      - \( i^4 + i^3 + i^2 + i + 1 \)
      - \( i^5 + i^4 + i^3 + i^2 + i + 1 \)
      - \( i^6 + i^5 + i^4 + i^3 + i^2 + i + 1 \)
      - \( i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1 \)
      - \( i^8 + i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1 \)

      d. Describe the pattern you observe, and algebraically prove your observation. In particular, compute

      \[ i^{195} + i^{194} + \ldots + i^2 + i + 1. \]

   b. Example: **Solution** (DOK 3)
      
      Joanne wants to graph a quadratic function whose roots are \( 5 \pm 2i \), and says:

      > I know the graph is a parabola, and the roots tell me that my function does not cross the x-axis, but I'm not sure where to go next -- how do I use this information to help with my graph?

      a. What can you deduce about the vertex of Joanne's parabola?

      b. With the information provided, can you graph Joanne's function? Why?

2. Use the relation \( i^2 = -1 \) and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers. **(N-CN.A.2) (DOK 1)**
a. Example: **Solution** (DOK 2)

Working with complex numbers allows us to solve equations like \( z^2 = -1 \) which cannot be solved with real numbers. Here we will investigate complex numbers which arise as square roots of certain complex numbers.

a. Find all complex square roots of -1, that is, find all numbers \( z = a + bi \) which satisfy \( z^2 = -1 \).

b. Find all complex square roots of 1.

c. Which complex numbers satisfy \( z^2 = i \)?

b. Example: **Solution** (DOK 2)

For each odd positive integer \( n \), the only real number solution to \( x^n = 1 \) is \( x = 1 \) while for even positive integers \( n \), \( x = 1 \) and \( x = -1 \) are solutions to \( x^n = 1 \). In this problem we look for all complex number solutions to \( x^n = 1 \) for some small values of \( n \).

a. Find all complex numbers \( a + bi \) whose cube is 1.

b. Find all complex numbers \( a + bi \) whose fourth power is 1.

c. Example: **Solution** (DOK 3)

a. Let \( z = 1 + i \) where \( i^2 = -1 \). Calculate \( z^2 \), \( z^3 \), and \( z^4 \).

b. Graph \( z \), \( z^2 \), \( z^3 \), and \( z^4 \) in the complex plane. What do you notice about the positions of these numbers?

c. What is \( z^{100} \)? Explain.

3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers. **(N-CN.A.3) (DOK 1)**

Represent complex numbers and their operations on the complex plane. **(N-CN.B)**

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number. **(N-CN.B.4) (DOK 1,2)**

5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example, \((-1 + \sqrt{3}i)^3 = 8 because (-1 + \sqrt{3}i) has modulus 2 and argument 120°*. **(N-CN.B.5) (DOK 1,2)**

   a. Example: **Solution** (DOK 2)
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints. (N-CN.B.6) (DOK 1)
   a. Example: Solution (DOK 3)
      
      In this problem, you will compute some values related to the complex numbers $2 + i$ and $5 - 3i$.
      
      a. Plot $2 + i$ and $5 - 3i$ in the complex plane.
      b. How far is $2 + i$ from $5 - 3i$? That is, what is the length of the line segment between $2 + i$ from $5 - 3i$?
      c. What is the modulus of the difference of $2 + i$ and $5 - 3i$?
      d. What is the midpoint of the line segment between $2 + i$ and $5 - 3i$?
      e. What is the average of $2 + i$ and $5 - 3i$?
      f. Describe, using the complex plane, the relationship between your answers in (b) and (c).
      g. Describe, using the complex plane, the relationship between your answers in (d) and (e).

Use complex numbers in polynomial identities and equations. (N-CN.C)

7. Solve quadratic equations with real coefficients that have complex solutions. (N-CN.C.7) (DOK 1)
   a. Example: Solution (DOK 3)
Renee reasons as follows to solve the equation $x^2 + x + 1 = 0$.

*First I will rewrite this as a square plus some number.*

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$  

*Now I can subtract $\frac{3}{4}$ from both sides of the equation*  

$$\left(x + \frac{1}{2}\right)^2 = -\frac{3}{4}.$$  

*But I can't take the square root of a negative number so I can't solve this equation.*

a. Show how Renee might have continued to find the complex solutions of $x^2 + x + 1 = 0$.

b. Apply Renee's reasoning to find the solutions to $x^2 + 4x + 6 = 0$.

8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$. *(N-CN.C.8) (DOK 1,2)*

   a. Example: [Solution](DOK 2)

   For each odd positive integer $n$, the only real number solution to $x^n = 1$ is $x = 1$ while for even positive integers $n$, $x = 1$ and $x = -1$ are solutions to $x^n = 1$. In this problem we look for all complex number solutions to $x^n = 1$ for some small values of $n$.

   a. Find all complex numbers $a + bi$ whose cube is 1.

   b. Find all complex numbers $a + bi$ whose fourth power is 1.

9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials. *(N-CN.C.9) (DOK 1,2)*
Vector and Matrix Quantities

Represent and model with vector quantities. (N-VM.A)

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., \( \mathbf{v} \), \( |\mathbf{v}| \), \( ||\mathbf{v}|| \), \( \mathbf{v} \)). (N-VM.A.1) (DOK 1)

2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point. (N-VM.A.2) (DOK 1)

3. (+) Solve problems involving velocity and other quantities that can be represented by vectors. (N-VM.A.3) (DOK 1, 2)

Perform operations on vectors. (N-VM.B)

4. (+) Add and subtract vectors.
   a. a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
   b. b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
   c. c. Understand vector subtraction \( \mathbf{v} - \mathbf{w} \) as \( \mathbf{v} + (-\mathbf{w}) \), where \(-\mathbf{w}\) is the additive inverse of \( \mathbf{w} \), with the same magnitude as \( \mathbf{w} \) and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise. (N-VM.B.4) (DOK 1, 2)

5. (+) Multiply a vector by a scalar.
   a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as \( c(v_x, v_y) = (cv_x, cv_y) \).
   b. Compute the magnitude of a scalar multiple \( c\mathbf{v} \) using \( ||c\mathbf{v}|| = |c| |\mathbf{v}| \). Compute the direction of \( c\mathbf{v} \) knowing that when \( |c|\mathbf{v} \neq 0 \), the direction of \( c\mathbf{v} \) is either along \( \mathbf{v} \) (for \( c > 0 \)) or against \( \mathbf{v} \) (for \( c < 0 \)). (N-VM.B.5) (DOK 1, 2)

Perform operations on matrices and use matrices in applications. (N-VM.C)

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network. (N-VM.C.6) (DOK 1, 2)

7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled. (N-VM.C.7) (DOK 1)

8. (+) Add, subtract, and multiply matrices of appropriate dimensions. (N-VM.C.8) (DOK 1)

9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties. (N-VM.C.9) (DOK 1)

10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse. (N-VM.C.10) (DOK 1)

11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors. (N-VM.C.11) (DOK 1, 2)

12. (+) Work with \( 2 \times 2 \) matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area. (N-VM.C.12) (DOK 1, 2)
Expressions. An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, \( p + 0.05p \) can be interpreted as the addition of a 5% tax to a price \( p \). Rewriting \( p + 0.05p \) as \( 1.05p \) shows that adding a tax is the same as multiplying the price by a constant factor.

Algebraic manipulations are governed by the properties of operations and exponents, and the conventions of algebraic notation. At times, an expression is the result of applying operations to simpler expressions. For example, \( p + 0.05p \) is the sum of the simpler expressions \( p \) and \( 0.05p \). Viewing an expression as the result of operation on simpler expressions can sometimes clarify its underlying structure.

A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.

Equations and inequalities. An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal. These values are the solutions to the equation. An identity, in contrast, is true for all values of the variables; identities are often developed by rewriting an expression in an equivalent form.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, but have a solution in a larger system. For example, the solution of \( x + 1 = 0 \) is an integer, not a whole number; the solution of \( 2x + 1 = 0 \) is a rational number, not an integer; the solutions of \( x^2 - 2 = 0 \) are real numbers, not rational numbers; and the solutions of \( x^2 + 2 = 0 \) are complex numbers, not real numbers.

The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, \( A = ((b_1 + b_2)/2)h \), can be solved for \( h \) using the same deductive process.

Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

Connections to Functions and Modeling. Expressions can define functions, and equivalent expressions define the same function. Asking when two functions have the same value for the same input leads to an equation; graphing
the two functions allows for finding approximate solutions of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling.
Algebra Overview

Seeing Structure in Expressions
- Interpret the structure of expressions
- Write expressions in equivalent forms to solve problems

Arithmetic with Polynomials and Rational Expressions
- Perform arithmetic operations on polynomials
- Understand the relationship between zeros and factors of polynomials
- Use polynomial identities to solve problems
- Rewrite rational expressions

Creating Equations
- Create equations that describe numbers or relationships

Reasoning with Equations and Inequalities
- Understand solving equations as a process of reasoning and explain the reasoning
- Solve equations and inequalities in one variable
- Solve systems of equations
- Represent and solve equations and inequalities graphically

Mathematical Practices
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
Seeing Structure in Expressions

Interpret the structure of expressions (A-SSE.A)

1. Interpret expressions that represent a quantity in terms of its context.*
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret \( P(1+r)^n \) as the product of \( P \) and a factor not depending on \( P \). (A-SSE.A.1) (DOK 1,2)

1. Example: Solution (DOK 3)
   Consider the algebraic expressions below:
   
   \[(n + 2)^2 - 4 \quad \text{and} \quad n^2 + 4n.\]

   a. Use the figures below to illustrate why the expressions are equivalent:

   b. Find some ways to algebraically verify the same result.

2. Example: Solution (DOK 3)
   A physics professor says: "Of course, it is easy to see that
   
   \[L_0 \sqrt{1 - \frac{v^2}{c^2}} = 0\]

   when \( v = c.\"

   a. Give a possible explanation in terms of the structure of the expression on the left why the professor might say that.

   b. Assuming that \( L_0 \) and \( c \) are positive, what is the greatest possible value of the expression on the left? Explain your answer in terms of the structure of the expression.

3. Example: Solution (DOK 3)
A landscaping company prepares two mixtures of fertilizer.

a. Mixture A contains 5 liters of liquid fertilizer and 50 liters of water. What is the concentration of fertilizer (by volume) in Mixture A?

b. Mixture B contains 10 liters of liquid fertilizer and 50 liters of water. What is the concentration of fertilizer (by volume) in Mixture B?

c. The volume of the fertilizer in Mixture B is twice the volume of the fertilizer in Mixture A. Why isn’t the concentration of fertilizer (by volume) in Mixture B double that of Mixture A?

d. Write an expression for the concentration of fertilizer (by volume) in Mixture A in terms of the volume of the fertilizer in Mixture A, \( F_a \), and the volume of the water, \( W \).

e. Explain your answer to (c) in terms of \( F_a \) and \( W \).

4. Example: **Solution** (DOK 3)

Consider the expression

\[
\frac{1}{R_1} + \frac{1}{R_2}
\]

where \( R_1 \) and \( R_2 \) are positive.

Suppose we increase the value of \( R_1 \) while keeping \( R_2 \) constant. Does the value of the expression above increase, decrease, or stay the same? Explain in terms of the structure of the expression.

2. Example: **Solution** (DOK 3)

The height (in feet) of a thrown horseshoe \( t \) seconds into flight can be described by the expression

\[
\frac{3}{16} + 18t - 16t^2.
\]

The expressions (a)-(d) below are equivalent. Which of them most clearly reveals the maximum height of the horseshoe's path? Explain your reasoning.

a. \( \frac{3}{16} + 18t - 16t^2 \)

b. \(-16\left(t - \frac{19}{16}\right)\left(t + \frac{1}{16}\right)\)

c. \( \frac{1}{16}(19 - 16t)(16t + 1) \)

d. \(-16\left(t - \frac{9}{16}\right)^2 + \frac{100}{16}\).
3. Example: Solution (DOK 2)
   Consider the expression
   \[
   \frac{R_1 + R_2}{R_1 R_2}
   \]
   where \( R_1 \) and \( R_2 \) are positive.

   Suppose we increase the value of \( R_1 \) while keeping \( R_2 \) constant. Find an equivalent expression whose structure makes clear whether the value of the expression increases, decreases, or stays the same.

4. Example: Solution (DOK 3)
   Give an explanation, in terms of the structure of the expression below, why it halves in value when \( n \) is quadrupled:
   \[
   \frac{e}{\sqrt{n}}.
   \]

5. Example: Solution (DOK 3)
   Fred has some colored kitchen floor tiles and wants to choose a pattern using them to make a border around white tiles. He generates patterns by starting with a row of four white tiles. He surrounds these four tiles with a border of colored tiles (Border 1). The design continues as shown below:

   ![Border Diagram]

   Fred writes the expression \( 4(b - 1) + 10 \) for the number of tiles in each border, where \( b \) is the border number, \( b \geq 1 \).

   a. Explain why Fred's expression is correct.

   b. Emma wants to start with five tiles in a row. She reasons, “Fred started with four tiles and his expression was \( 4(b - 1) + 10 \). So if I start with five tiles, the expression will be \( 5(b - 1) + 10 \). Is Emma's statement correct? Explain your reasoning.

   c. If Emma starts with a row of \( n \) tiles, what should the expression be?

6. Example: Solution (DOK 2)
A candy shop sells a box of chocolates for $30. It has $29 worth of chocolates plus $1 for the box. The box includes two kinds of candy: caramels and truffles. Lita knows how much the different types of candies cost per pound and how many pounds are in a box. She said,

*If \( x \) is the number of pounds of caramels included in the box and \( y \) is the number of pounds of truffles in the box, then I can write the following equations based on what I know about one of these boxes:*

* *\( x + y = 3 \)*
* *\( 8x + 12y + 1 = 30 \)*

Assuming Lita used the information given and her other knowledge of the candies, use her equations to answer the following:

a. How many pounds of candy are in the box?

b. What is the price per pound of the caramels?

c. What does the term \( 12y \) in the second equation represent?

d. What does \( 8x + 12y + 1 \) in the second equation represent?

7. Example: **Solution** (DOK 3)

Most savings accounts advertise an annual interest rate, but they actually compound that interest at regular intervals during the year. That means that, if you own an account, you'll be paid a portion of the interest before the year is up, and, if you keep that payment in the account, you'll start earning interest on the interest you've already earned.

For example, suppose you put $500 in a savings account that advertises 5% annual interest. If that interest is paid once per year, then your balance \( B \) after \( t \) years could be computed using the equation \( B = 500(1.05)^t \), since you'll end each year with 100% + 5% of the amount you began the year with.

On the other hand, if that same interest rate is compounded monthly, then you would compute your balance after \( t \) years using the equation

\[
B = 500 \left(1 + \frac{0.05}{12}\right)^{12t}
\]

a. Why does it make sense that the equation includes the term \( \frac{0.05}{12} \)? That is, why are we dividing 0.05 by 12?

b. How does this equation reflect the fact that you opened the account with $500?

c. What do the numbers 1 and \( \frac{0.05}{12} \) represent in the expression \( \left(1 + \frac{0.05}{12}\right)^{12t} \)?

d. What does the “12\( t \)” in the equation represent?

8. Example: **Solution** (DOK 3)
Suppose $P$ and $Q$ give the sizes of two different animal populations, where $Q > P$. In (a)-(f), say which of the given pair of expressions is larger. Briefly explain your reasoning in terms of the two populations.

a. $P + Q$ and $2P$

b. $\frac{P}{P + Q}$ and $\frac{P + Q}{2}$

c. $(Q - P)/2$ and $Q - P/2$

d. $P + 50t$ and $Q + 50t$

e. $\frac{P}{P + Q}$ and $0.5$

f. $\frac{P}{Q}$ and $\frac{Q}{P}$

9. Example: Solution (DOK 2)

A company uses two different-sized trucks to deliver sand. The first truck can transport $x$ cubic yards, and the second $y$ cubic yards. The first truck makes $S$ trips to a job site, while the second makes $T$ trips. What quantities do the following expressions represent in terms of the problem’s context?

a. $S + T$

b. $x + y$

c. $xS + yT$

d. $\frac{xS + yT}{S + T}$

10. Example: Solution (DOK 2)
A company uses two different-sized trucks to deliver concrete blocks. The first truck can transport \( x \) blocks per trip, and the second can transport \( y \) blocks per trip. The first truck makes \( S \) trips to a job site, while the second makes \( T \) trips. What do the following expressions represent in the context of the problem? Use the drop-down menus to construct a two-part description of the expression.

a. \( S + T \)

b. \( xS + yT \)

c. \( \frac{xS + yT}{S + T} \)

11. Example: \textbf{Solution} (DOK 2)

The profit, \( P \) (in thousands of dollars), that a company makes selling an item is a quadratic function of the price, \( x \) (in dollars), that they charge for the item. The following expressions for \( P(x) \) are equivalent:

\[
P(x) = -2x^2 + 24x - 54 \\
P(x) = -2(x - 3)(x - 9) \\
P(x) = -2(x - 6)^2 + 18
\]

1. Which of the equivalent expressions for \( P(x) \) reveals \textit{the price which gives a profit of zero} without changing the form of the expression?

2. Find a price which gives a profit of zero.

3. Which of the equivalent expressions for \( P(x) \) reveals \textit{the profit when the price is zero} without changing the form of the expression?

4. Find the profit when the price is zero.

5. Which of the equivalent expressions for \( P(x) \) reveals \textit{the price which produces the highest possible profit} without changing the form of the expression?

6. Find the price which gives the highest possible profit.
For each of parts (a), (c), and (e), students choose which form of the function is displayed, and then use that to answer the corresponding questions in parts (b), (d), and (f) respectively. Students are assessed both on selecting the correct form of the function and on their numerical answer for each part.

2. Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. (A-SSE.A.2) (DOK 1,2)

   a. Example: Solution (DOK 3)

      Consider the algebraic expressions below:

      $$ (n + 2)^2 - 4 \quad \text{and} \quad n^2 + 4n. $$

      a. Use the figures below to illustrate why the expressions are equivalent:

      ![Image of figures]

      b. Find some ways to algebraically verify the same result.

   b. Example: Solution (DOK 2)

      Find a value for $a$, a value for $k$, and a value for $n$ so that

      $$ (3x + 2)(2x - 5) = ax^2 + kx + n. $$

   c. Example: Solution (DOK 3)
A function \( f \) defined for \(-a < x < a\) is even if \( f(-x) = f(x) \) and is odd if \( f(-x) = -f(x) \) when \(-a < x < a\). In this task we assume \( f \) is defined on such an interval, which might be the full real line (i.e., \( a = \infty \)).

a. Show that \( f(x) = x^2 \) is even and \( g(x) = x^3 \) is odd.

b. Write \( f(x) = 3x^3 + 2x^2 - 5x + 7 \) as a sum \( f(x) = e(x) + o(x) \), where \( e \) is even and \( o \) is odd.

c. Do the same for the function \( f(x) = \frac{1}{1 - x} \) on the domain \(-1 < x < 1\). [Hint: multiply numerator and denominator by \( 1 + x \).]

d. Parts (b) and (c) suggest that it might always be possible to write \( f(x) = e(x) + o(x) \) where \( e \) is even and \( o \) is odd. Suppose that this is so, and use the definition of even and odd to write an equation expressing \( f(-x) \) in terms of \( e(x) \) and \( o(x) \).

e. You now have two equations: \( f(x) = e(x) + o(x) \) and the other one you obtained in part (d). Solve this system of equations for \( e(x) \) and \( o(x) \), and show that the resulting \( e(x) \) is even and the resulting \( o(x) \) is odd.

f. Based on your work in part (e), is it true or is it false that every function defined on the interval \(-a < x < a\) can be expressed as a sum of an even function and an odd function? Why?

g. Use your answer to part (e) to express \( f(x) = e^x \) as a sum of an even function and an odd function.

d. Example: Solution (DOK 3)

Suppose \( P \) and \( Q \) give the sizes of two different animal populations, where \( Q > P \). In (a)-(f), say which of the given pair of expressions is larger. Briefly explain your reasoning in terms of the two populations.

a. \( P + Q \) and \( 2P \)

b. \( \frac{P}{P + Q} \) and \( \frac{P + Q}{2} \)

c. \( (Q - P)/2 \) and \( Q - P/2 \)

d. \( P + 50t \) and \( Q + 50t \)

e. \( \frac{P}{P + Q} \) and 0.5

f. \( \frac{P}{Q} \) and \( \frac{Q}{P} \)

e. Example: Solution (DOK 2)
Rewrite each of the following expressions involving complex numbers in the form $a + bi$ where $a$ and $b$ are real numbers.

\[ a. \ (3 + 2i)(2 - 5i) \]
\[ b. \ (5 + 4i)(17 - 13i) - (5 + 3i)(17 - 13i) \]
\[ c. \ \left( \frac{5}{2} + \frac{7i}{2} \right)^2 - \left( \frac{5}{2} + \frac{i}{2} \right)^2 \]
\[ d. \ (1 + i)(13 - 4i)(1 - i) \]
\[ e. \ 1 + i + i^2 + i^3 \]

f. Example: Solution (DOK 2)

Let $a$ and $b$ be real numbers with $a > b > 0$ and $\frac{a^3 - b^3}{(a-b)^3} = \frac{73}{3}$. What is $\frac{b}{a}$?

g. Example: Determine whether each expression is equivalent to $(x^3 + 8)$. Select Yes or No for each expression.

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Write expressions in equivalent forms to solve problems (A-SSE.B)

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

a. Factor a quadratic expression to reveal the zeros of the function it defines.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. For example the expression $1.15^t$ can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate
equivalent monthly interest rate if the annual rate is 15%. \((A\text{-SSE.B.3})\) \((DOK 1,2)\)

1. Example: Solution \((DOK 2)\)

Consider the expression

\[
\frac{R_1 + R_2}{R_1R_2}
\]

where \(R_1\) and \(R_2\) are positive.

Suppose we increase the value of \(R_1\) while keeping \(R_2\) constant. Find an equivalent expression whose structure makes clear whether the value of the expression increases, decreases, or stays the same.

2. Example: Solution \((DOK 3)\)

a. Graph these equations on your graphing calculator at the same time. What happens? Why?

\[
\begin{align*}
y_1 &= (x-3)(x+1) \\
y_2 &= x^2-2x-3 \\
y_3 &= (x-1)^2-4 \\
y_4 &= x^2-2x+1
\end{align*}
\]

b. Below are the first three equations from the previous problem.

\[
\begin{align*}
y_1 &= (x-3)(x+1) \\
y_2 &= x^2-2x-3 \\
y_3 &= (x-1)^2-4
\end{align*}
\]

These three equations all describe the same function. What are the coordinates of the following points on the graph of the function? From which equation is each point most easily determined? Explain.

i. vertex: ___

ii. \(y\)-intercept: ___

iii. \(x\)-intercept(s): ___

c. Make up an equation for a quadratic function whose graph satisfies the given condition. Use whatever form is most convenient.

i. Has a vertex at \((-2, -5)\).

ii. Has a \(y\)-intercept at \((0, 6)\)

iii. Has \(x\)-intercepts at \((3, 0)\) and \((5, 0)\)

iv. Has \(x\)-intercepts at the origin and \((4, 0)\)

v. Goes through the points \((4, 2)\) and \((1, 2)\)

3. Example: Solution \((DOK 3)\)
The profit that a company makes selling an item (in thousands of dollars) depends on the price of the item (in dollars). If $p$ is the price of the item, then three equivalent forms for the profit are:

- Standard form: $-2p^2 + 24p - 54$
- Factored form: $-2(p - 3)(p - 9)$
- Vertex form: $-2(p - 6)^2 + 18$

Which form is most useful for finding

a. The prices that give a profit of zero dollars?

b. The profit when the price is zero?

c. The price that gives the maximum profit?

4. Example: Solution (DOK 2)

After a container of ice cream has been sitting in a room for $t$ minutes, its temperature in degrees Fahrenheit is

$$a - b2^{-t} + b,$$

where $a$ and $b$ are positive constants. Write this expression in a form that

a. Shows that the temperature is always less than $a + b$.

b. Shows that the temperature is never less than $a$.

5. Example: Solution (DOK 2)

Let $a$ and $b$ be real numbers with $a > b > 0$ and $\frac{a^3 - b^3}{(a - b)^3} = \frac{28}{3}$. What is $\frac{b}{a}$?

6. Example: Solution (DOK 3)

Four physicists describe the amount of a radioactive substance, $Q$ in grams, left after $t$ years:

a. $Q = 300e^{-0.0577t}$

b. $Q = 300(1/2)^{t/12}$

c. $Q = 300 \cdot 0.9439^t$

d. $Q = 252.290 \cdot 0.9439^{t-3}$

(i) Show that the expressions describing the radioactive substance are all equivalent (using appropriate rounding).

(ii) What aspect of the decay of the substance does each of the formulas highlight?
7. Example: **Solution** (DOK 2)

The profit, $P$, in thousands of dollars, that a company makes selling an item is a quadratic function of the price, $x$ (in dollars), that they charge for the item. The following expressions for $P(x)$ are equivalent:

$$P(x) = -2x^2 + 24x - 54$$
$$P(x) = -2(x - 3)(x - 9)$$
$$P(x) = -2(x - 6)^2 + 18$$

1. Which of the equivalent expressions for $P(x)$ reveals *the price which gives a profit of zero* without changing the form of the expression?

2. Find a price which gives a profit of zero.

3. Which of the equivalent expressions for $P(x)$ reveals *the profit when the price is zero* without changing the form of the expression?

4. Find the profit when the price is zero.

5. Which of the equivalent expressions for $P(x)$ reveals *the price which produces the highest possible profit* without changing the form of the expression?

6. Find the price which gives the highest possible profit.

*For each of parts (a), (c), and (e), students choose which form of the function is displayed, and then use that to answer the corresponding questions in parts (b), (d), and (f) respectively. Students are assessed both on selecting the correct form of the function and on their numerical answer for each part.*

8. Example: **Solution** (DOK 3)

   a. What is the minimum value taken by the expression $(x - 4)^2 + 6$? How does the structure of the expression help to see why?

   b. Rewrite the quadratic expression $x^2 - 6x - 3$ in the form $(x - _)_2 + _$ and find its minimum value.

   c. Rewrite the quadratic expression $-2x^2 + 4x + 3$ in the form $(_)(x - _)_2 + _$. What is its maximum value? Explain how you know.

9. Example: **Solution** (DOK 3)
Joanne wants to graph a quadratic function whose roots are $5 \pm 2i$, and says:

*I know the graph is a parabola, and the roots tell me that my function does not cross the $x$-axis, but I'm not sure where to go next -- how do I use this information to help with my graph?*

a. What can you deduce about the vertex of Joanne's parabola?

b. With the information provided, can you graph Joanne's function? Why?

10. **Example: Solution (DOK 2)**

   The figure shows a graph of the function $f(x) = x^2$.

   ![Graph of $f(x) = x^2$](image)

   a) For each of the graphs of quadratic functions below, find values of $a$, $b$, and $c$ so that the function $f(x) = ax^2 + bx + c$ has that graph. (For example, the graph in the first part corresponds to $a = 1$, $b = 0$, and $c = 0$).

   ![Graphs of quadratic functions](image)
4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* (A-SSE.B.4) (DOK 1,2,3)

a. Example: Solution (DOK 2)

b) For each of the following three descriptions of graphs of quadratic functions, sketch a graph by hand, and then find a function in the form \( f(x) = ax^2 + bx + c \) whose graph fits the description.

- The graph is concave down and has its vertex at \((0, 7)\).
- The graph is concave up and has its vertex at \((-7, 0)\).
- The graph has its vertex at \((4, -9)\) and passes through the point \((6, -1)\).

c) If the graph of the function \( f(x) = ax^2 + bx + c \) is as below, determine the signs of \(a\), \(b\), and \(c\).
Consider the picture below, consisting of a nested sequence of five equilateral triangles, colored in black. Each of the black triangles is made by connecting the three midpoints of the sides of the immediately larger white triangle.

Find and evaluate a sum to compute the total area of the black region (that is, the sum of the areas of the five black triangles) given that the largest triangle in the diagram has area 1.

b. Example: Solution (DOK 3)
Susan has an ear infection. The doctor prescribes a course of antibiotics. Susan is told to take 250 mg doses of the antibiotic regularly every 12 hours for 20 days.

Susan is curious and wants to know how much of the drug will be in her body over the course of the 20 days. She does some research online and finds out that at the end of 12 hours, about 4% of the drug is still in the body.

a. What quantity of the drug is in the body right after the first dose, the second dose, the third dose, the fourth dose?

b. When will the total amount of the antibiotic in Susan's body be the highest? What is that amount?

c. Answer Susan's original question: Describe how much of the drug will be in her body at various points over the course of the 20 days.

c. Example: Solution (DOK 2)
In this task we will investigate an interesting mathematical object called the Cantor Set. It is a simple example of a fractal with some pretty weird properties. Here is how we construct it: Draw a black interval on the number line from 0 to 1 and call this set $C_0$. Create a new set called $C_1$ by removing the “middle third” of the interval, i.e. all the numbers between $\frac{1}{3}$ and $\frac{2}{3}$. So $C_1$ consists of two black intervals of length one-third. We continue by removing the middle third of each of the two remaining intervals and calling this set $C_2$. We then take out the middle third of each remaining black interval to create $C_3$ and so on (see diagram below).

![Diagram of Cantor Set]

- **a.** How many black intervals are in $C_0$, $C_1$, $C_2$, $C_3$, \ldots, $C_{10}$?
- **b.** Add up the total length of the pieces that are removed at each stage. We can think of this as the total length of the “gaps” between the black intervals. What is the total length of the gaps in $C_0$, $C_1$, $C_2$, $C_3$, \ldots, $C_{10}$?
- **c.** If we continued to remove more and more middle thirds, how much length of the original interval would we eventually remove? Are there any points that would be left?

*Note: As a thought experiment, imagine we could continue the middle third removal forever. There are still some numbers that would never get removed like 0, $\frac{1}{3}$, $\frac{2}{3}$, 1 etc. The set of numbers that will never be removed is called the Cantor Set and it has some amazing properties. For example, there are infinitely many numbers in the Cantor Set (even uncountably many numbers), but it contains no intervals of numbers and its total length is zero.*

- **d.** Example: **Solution** (DOK 3)
For 70 years, Oseola McCarty earned a living washing and ironing other people’s clothing in Hattiesburg, Mississippi. Although she did not earn much money, she budgeted her money wisely, lived within her means, and began saving at a very young age. Before she died, she drew worldwide attention by donating $150,000 to the University of Southern Mississippi for a scholarship fund in her name. The fact that Ms. McCarty was able to save so much money and generously gave it away is an inspiration to many others. She was honored with the Presidential Citizens Medal for her generosity. How did she do it?

Let’s assume that she saved the same amount at the end of each year and invested it in a savings account earning 5% per year compounded annually. (When you contribute the same amount each year to an account it is called an annuity.) How much do you think Ms. McCarty would have to save each year in order to accumulate $150,000 over a 70-year period?

a. Before we figure it out, take a guess.
   • $100
   • $250
   • $500
   • $1,000
   • $2,000
b. Suppose Ms. McCarty saved $100 and then deposited it at the end of the year in an account that earns 5% interest, compounded annually.
    • How much will it be worth at the end of the second year? At the end of the third year? At the end of the 70th year?
    • Write an expression that represents the value of an investment of \( C \) dollars after 70 years. Assume as above that it is deposited at the end of the first year in an account that earns 5% interest, compounded annually.

c. Now suppose Ms. McCarty saved another $100 in the second year and then deposited it at the end of that year in her account.
    • How much will it be worth at the end of the third year? At the end of the fourth year? At the end of the 70th year?
    • Write an expression that represents the value of an investment of \( C \) dollars after 69 years.

d. Suppose Ms. McCarty saved $100 each and every year for 70 years. Each time, she deposited it in her account at the end of the year.
    • How much would she have saved? What would it be worth at the end of 70 years?
    • Write an expression that represents the value of an investment of \( C \) dollars deposited each year for 70 years. Assume as above that it is always deposited at the end of the year in an account that earns 5% interest, compounded annually.

e. Had she saved $1,000 a year, how much would she have had after 70 years under the same conditions?

f. How much would she have to save each year in order to accumulate $150,000 after 70 years? How does this compare to your guess? Are you surprised by the answer?

g. The future value \( FV \) of an annuity is the total value of the annuity after a certain number of years. The formula for the future value of an annuity is shown below.

\[
FV = C \cdot \left[ \frac{(1 + r)^t - 1}{r} \right]
\]

Based on the work you did above, what is the meaning of \( C \) in this context? What is the meaning of \( r \) in this context? What is the meaning of \( t \) in this context?

e. Example: Solution (DOK 3)
Michelle, Hillary, and Cory created a YouTube video, and have a plan to get as many people to watch it as possible. They will each share the video with 3 of their best friends, and create a caption on the video that says, "Please share this video with 3 of your best friends." Each time the video is shared with someone, that person instantly views the video only once and sends it to exactly 3 more people. In addition, assume that every person only receives the video once.

Suppose that after 1 hour since the video was posted, Michelle, Hillary, and Cory have watched the video; after 2 hours, all of their friends have watched the video; and so on.

a. Fill in the table for the number of people who received the video during the given hour. (Assume that Michelle, Hillary, and Cory "received" the video as well.)

<table>
<thead>
<tr>
<th>Hour</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>10</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>People Receiving Video</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. What is the total number of views the video has after hour 3? After hour 4? After hour 10? After hour n?

c. Let's look at only the people who have viewed the video via Hillary. Fill in the following table for the number of people who have received the video via Hillary in the given hour.

<table>
<thead>
<tr>
<th>Hour</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>10</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>People Receiving Video</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

d. How many total people have seen the video via Hillary after hour n?

e. Using the generalized (hour n) expressions you developed in parts (b) and (d), give an expression, in terms of n, for the number of people who viewed the video not via Hillary after hour n.

f. Let H represent your answer to part (d). In terms of H, how many people viewed the video via Cory and Michelle after hour n?

g. Using your answers in questions (e) and (f), write a formula (not a sum) in terms of n for the number of people who have viewed the video via Hillary after hour n.

h. Using this formula, determine the total number of views the video will have after 10 hours.

i. Suppose that instead of 3 friends creating a video and sending it to their 3 best friends, we started off with r friends creating a video and each sending it to r of their best friends. Write a formula (not a sum) for the number of views the video will have after n hours.
Perform arithmetic operations on polynomials (A-APR.A)

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. (A-APR.A.1) (DOK 1)
   e. Example: Solution (DOK 2)

   Felicia notices what appears to be an interesting pattern between powers of 11 and powers of \( x + 1 \):

   \[
   
   \begin{align*}
   11^0 &= 1 & (x + 1)^0 &= 1 \\
   11^1 &= 11 & (x + 1)^1 &= x + 1 \\
   11^2 &= 121 & (x + 1)^2 &= x^2 + 2x + 1
   \end{align*}
   
   The digits of the number \( 11^n \) are the same as the coefficients of the polynomial \( (x + 1)^n \). Is this always true?

   a. Does this pattern continue for \( n = 3 \) and \( n = 4 \)?
   b. What is the answer to Felicia’s question?

   f. Example: Solution (DOK 3)

   A non-negative polynomial \( f \) is a polynomial which never takes negative values, that is, \( f(x) \geq 0 \) for all real values of \( x \).

   a. Decide which of the following polynomials are non-negative:

   \[
   x^2 \quad x^2 - 1 \quad x^3 \quad 100000 - x^2 \quad mx + b
   \]

   In the last part, consider various possibilities for \( m \) and \( b \).

   b. Show that if \( g \) is a polynomial, then \( g^2 \) is a non-negative polynomial. Use this fact to generate some non-negative polynomials.

   c. Are all of the coefficients of a non-negative polynomial necessarily positive?

   d. Is there a non-negative polynomial which has all negative coefficients?

   e. Find a non-negative polynomial which is not the square of another polynomial.

   g. Example: Multiply and combine like terms to determine the product of these polynomials.

   \((2x - 3)(5x + 6)\)
Understand the relationship between zeros and factors of polynomials (A-APR.B)

2. Know and apply the Remainder Theorem: For a polynomial \( p(x) \) and a number \( a \), the remainder on division by \( x - a \) is \( p(a) \), so \( p(a) = 0 \) if and only if \( (x - a) \) is a factor of \( p(x) \). (A-APR.B.2) (DOK 1,2)

a. Example: **Solution** (DOK 2)

Consider the polynomial function

\[
P(x) = x^4 - 3x^3 + ax^2 - 6x + 14,
\]

where \( a \) is an unknown real number. If \((x - 2)\) is a factor of this polynomial, what is the value of \( a \)?

b. Example: **Solution** (DOK 3)

Suppose \( f \) is a quadratic function given by the equation

\[
f(x) = ax^2 + bx + c \]

where \( a, b, c \) are real numbers and we will assume that \( a \) is non-zero.

a. If \( 0 \) is a root of \( f \) explain why \( c = 0 \) or, in other words, \( ax^2 + bx + c \) is evenly divisible by \( x \).

b. If \( 1 \) is a root of \( f \) explain why \( ax^2 + bx + c \) is evenly divisible by \( x - 1 \).

c. Suppose \( r \) is a real number. If \( r \) is a root of \( f \) explain why \( ax^2 + bx + c \) is evenly divisible by \( x - r \).

c. Example: **Solution** (DOK 3)

Suppose \( p(x) \) is a polynomial of degree \( d > 0 \).

a. If \( p(0) = 0 \), show that \( p(x) \) is evenly divisible by \( x \).

b. If \( p(1) = 0 \), show that \( p(x) \) is evenly divisible by \( x - 1 \).

c. If \( r \) is a real number such that \( p(r) = 0 \), show that \( p(x) \) is evenly divisible by \( x - r \).

d. Using part (c) show that \( p \) can have at most \( d \) distinct roots, that is, there can be at most \( d \) numbers \( r_1, \ldots, r_d \) with \( p(r_1) = \cdots = p(r_d) = 0 \).
d. Example: Solution (DOK 3)

Suppose \( f \) is a quadratic function given by the equation
\[ f(x) = ax^2 + bx + c \]
where \( a, b, c \) are real numbers and \( a \) is non-zero.

a. Explain why \( f \) can have at most two roots; that is explain why there can be at most two distinct real numbers \( r_1, r_2 \) so that
\[ f(r_1) = f(r_2) = 0. \]
b. Give examples to show that it is possible for \( f \) to have zero, one, or two real roots.

e. Example: Solution (DOK 3)

a. Sketch graphs of the functions \( f \) and \( F \) given by \( f(x) = |x| \) and
\[ F(x) = x^2 \]
for \(-2 \leq x \leq 2\).

b. Suppose \( g \) is the function given by \( g(x) = \frac{f(x)}{x} \) for \( x \neq 0 \) and \( G \) is the function given by \( G(x) = \frac{F(x)}{x} \) for \( x \neq 0 \). Sketch graphs of the functions \( g \) and \( G \) for \( x \neq 0 \) and \(-2 \leq x \leq 2\).

c. Is there a natural way to define \( g \) and \( G \) when \( x = 0 \)? Explain.

f. Example: Solution (DOK 3)

Mike is trying to sketch a graph of the polynomial
\[ f(x) = x^3 + 4x^2 + x - 6. \]

He notices that the coefficients of \( f(x) \) add up to zero
\[ 1 + 4 + 1 - 6 = 0 \]
and says

This means that 1 is a root of \( f(x) \), and I can use this to help factor \( f(x) \) and produce the graph.

a. Is Mike right that 1 is a root of \( f(x) \)? Explain his reasoning.
b. Find all roots of \( f(x) \).
c. Find all inputs \( x \) for which \( f(x) < 0 \).
d. Use the information you have gathered to sketch a rough graph of \( f \).

3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial. (A-APR.B.3) (DOK 1,2)

a. Example: Solution (DOK 3)
Mike is trying to sketch a graph of the polynomial

\[ f(x) = x^3 + 4x^2 + x - 6. \]

He notices that the coefficients of \( f(x) \) add up to zero \((1 + 4 + 1 - 6 = 0)\) and says

\[ \text{This means that 1 is a root of } f(x), \text{ and I can use this to help factor } f(x) \text{ and produce the graph.} \]

a. Is Mike right that 1 is a root of \( f(x) \)? Explain his reasoning.

b. Find all roots of \( f(x) \).

c. Find all inputs \( x \) for which \( f(x) < 0 \).

d. Use the information you have gathered to sketch a rough graph of \( f \).

b. Example: Solution (DOK 3)

a. Find all the values of \( x \) for which the equation \( 9x = x^3 \) is true.

b. Use graphing technology to graph \( f(x) = x^3 - 9x \). Explain where you can see the answers from part (a) in this graph, and why.

c. Someone attempts to solve \( 9x = x^3 \) by dividing both sides by \( x \), yielding \( 9 = x^2 \), and going from there. Does this approach work? Why or why not?

c. Example: Solution (DOK 3)

Graph the functions given by the equations \( y = (x - 1)(x + 2)(x - 5) \) and \( y = 3(x - 1)(x + 2)(x - 5) \) with a viewing window \(-10 \leq x \leq 10 \) and \(-100 \leq y \leq 100 \).

a. Describe any similarities you see between the two graphs, and explain how you can see those similarities in the given equations.

b. Write an equation for a function whose graph in the \( xy \)-plane has \( x \)-intercepts at \(-9, -6, 0, \) and \( 4 \). Graph your equation to verify that it works.

d. Example: Solution (DOK 3)
Use polynomial identities to solve problems (A-APR.C)

4. Prove polynomial identities and use them to describe numerical relationships. For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples. (A-APR.C.4) (DOK 1,2,3)
   a. Example: Solution (DOK 3)
Alice was having a conversation with her friend Trina, who had a discovery to share:

“Pick any two integers. Look at the sum of their squares, the difference of their squares, and twice the product of the two integers you chose. Those three numbers are the sides of a right triangle.”

Trina had tried this several times and found that it worked for every pair of integers she tried. However, she admitted that she wasn’t sure whether this "trick" always works, or if there might be cases in which the trick doesn’t work.

a. Investigate Trina’s conjecture for several pairs of integers. Does her trick appear to work in all cases, or only in some cases?

b. If Trina’s conjecture is true, then give a precise statement of the conjecture, using variables to represent the two chosen integers, and prove it. If the conjecture is not true, modify it so that it is a true statement, and prove the new statement.

c. Use Trina’s trick to find an example of a right triangle in which all of the sides have integer length, all three sides are longer than 100 units, and the three side lengths do not have any common factors.

5. (+) Know and apply the Binomial Theorem for the expansion of \((x + y)^n\) in powers of \(x\) and \(y\) for a positive integer \(n\), where \(x\) and \(y\) are any numbers, with coefficients determined for example by Pascal’s Triangle.1

(A-APR.C.5) (DOK 1,2)

a. Example: Solution (DOK 2)

Felicia notices what appears to be an interesting pattern between powers of 11 and powers of \(x + 1\):

\[
\begin{align*}
11^0 &= 1 \\
11^1 &= 11 \\
11^2 &= 121 \\
(x + 1)^0 &= 1 \\
(x + 1)^1 &= x + 1 \\
(x + 1)^2 &= x^2 + 2x + 1
\end{align*}
\]

The digits of the number \(11^n\) are the same as the coefficients of the polynomial \((x + 1)^n\). Is this always true?

a. Does this pattern continue for \(n = 3\) and \(n = 4\)?

b. What is the answer to Felicia’s question?

1 The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.
Rewrite rational expressions (A-APR.D)

6. Rewrite simple rational expressions in different forms; write \( a(x)/b(x) \) in the form \( q(x) + \frac{r(x)}{b(x)} \), where \( a(x), b(x), q(x), \) and \( r(x) \) are polynomials with the degree of \( r(x) \) less than the degree of \( b(x) \), using inspection, long division, or, for the more complicated examples, a computer algebra system. (A-APR.D.6) (DOK 1,2)
   a. Example: Solution (DOK 2)

The US Department of Energy keeps track of fuel efficiency for all vehicles sold in the United States. Each car has two fuel economy numbers, one measuring efficiency for city driving and one for highway driving. For example, a 2012 Volkswagen Jetta gets 29.0 miles per gallon (mpg) in the city and 39.0 mpg on the highway.

Many banks have "green car loans" where the interest rate is lowered for loans on cars with high combined fuel economy. This number is not the average of the city and highway economy values. Rather, the combined fuel economy (as defined by the federal Corporate Average Fuel Economy standard) for \( x \) mpg in the city and \( y \) mpg on the highway, is computed as

\[
\text{combined fuel economy} = \frac{1}{\frac{1}{x} + \frac{1}{y}}.
\]

a. What is the combined fuel economy for the 2012 Volkswagen Jetta? Give your answer to one decimal place.

b. For most conventional cars, the highway fuel economy is 10 mpg higher than the city fuel economy. If we set the city fuel economy to be \( x \) mpg for such a car, what is the combined fuel economy in terms of \( x \)? Write your answer as a single rational function, \( a(x)/b(x) \).

c. Rewrite your answer from (b) in the form of \( q(x) + \frac{r(x)}{b(x)} \) where \( q(x), r(x) \) and \( b(x) \) are polynomials and the degree of \( r(x) \) is less than the degree of \( b(x) \).

d. Use your answer in (c) to conclude that if the city fuel economy, \( x \), is large, then the combined fuel economy is approximately \( x + 5 \).

b. Example: Solution (DOK 3)
Ancient Egyptians used unit fractions, such as $\frac{1}{2}$ and $\frac{1}{3}$, to represent all other fractions. For example, they might express the number $\frac{3}{4}$ as $\frac{1}{2} + \frac{1}{4}$. The Egyptians did not use a given unit fraction more than once so they would not have written $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$.

a. How might the Egyptians have expressed the number $\frac{5}{7}$? What about $\frac{5}{13}$?

b. We will see how we can use identities between rational expressions to help in our understanding of Egyptian fractions. Verify the following identity for any $x > 0$:

$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)}$$

c. Show that each unit fraction $\frac{1}{n}$, with $n \geq 2$, can be written as a sum of two or more different unit fractions.

d. Describe a procedure for writing any positive rational number, say $\frac{p}{q}$ with $p, q > 0$, as an Egyptian fraction.

7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions. (A-APR.D.7) (DOK 1)
Creating Equations*  

Create equations that describe numbers or relationships (A-CED.A)

**Example:** A rectangular garden measures 13 meters by 17 meters and has a cement walkway around its perimeter, as shown. The width of the walkway remains constant on all four sides. The garden and walkway have a combined area of 396 square meters.

![Diagram of garden and walkway]

**Part A**
Write an equation that could be used to help determine the width, \( w \), of the walkway.

**Part B**
Determine the width, in meters, of the walkway.

<table>
<thead>
<tr>
<th>Item</th>
<th>Claim</th>
<th>Domain</th>
<th>Target</th>
<th>DOK</th>
<th>CONTENT</th>
<th>MP</th>
<th>Key</th>
</tr>
</thead>
<tbody>
<tr>
<td>#13</td>
<td>2</td>
<td>A-CED</td>
<td>A</td>
<td>2</td>
<td>HSA.CED.A</td>
<td>2, 7</td>
<td>See Below</td>
</tr>
</tbody>
</table>

**Exemplar:**
First response box: \((17 + 2w)(13 + 2w) = 396\) or an equivalent equation

Second response box: \(\frac{s}{2}\) or \(w = \frac{5}{2}\) or equivalent values

**Rubric:**
(2 points) Student enters a correct equation in Part A and a correct width in Part B.

(1 point) Student enters a correct equation in Part A or a correct width in Part B.

1. Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* (A-CED.A.1) (DOK 1,2)
   a. Example: [Solution] (DOK 2)

   A team of farm-workers was assigned the task of harvesting two fields, one twice the size of the other. They worked for the first half of the day on the larger field. Then the team split into two groups of equal number. The first group continued working in the larger field and finished it by evening. The second group harvested the smaller field, but did not finish by evening. The next day one farm-worker finished the smaller field in a single day's work. How many farm-workers were on the team?

   b. Example: [Solution] (DOK 2)
A government buys $x$ fighter planes at $z$ dollars each, and $y$ tons of wheat at $w$ dollars each. It spends a total of $B$ dollars, where $B = xz + yw$. In (a)-(c), write an equation whose solution is the given quantity.

a. The number of tons of wheat the government can afford to buy if it spends a total of $100$ million, wheat costs $300$ per ton, and it must buy 5 fighter planes at $15$ million each.

b. The price of fighter planes if the government bought 3 of them, in addition to 10,000 tons of wheat at $500$ a ton, for a total of $50$ million.

c. The price of a ton of wheat, given that a fighter plane costs 100,000 times as much as a ton of wheat, and that the government bought 20 fighter planes and 15,000 tons of wheat for a total cost of $90$ million.

c. Example: Solution (DOK 2)

A checking account is set up with an initial balance of $4800$, and $400$ is removed from the account each month for rent (no other transactions occur on the account).

a. Write an equation whose solution is the number of months, $m$, it takes for the account balance to reach $2000$.

b. Make a plot of the balance after $m$ months for for $m = 1, 3, 5, 7, 9, 11$ and indicate on the plot the solution to your equation in part (a).

d. Example: Solution (DOK 2)

Suppose a friend tells you she paid a total of $16,368$ for a car, and you’d like to know the car’s list price (the price before taxes) so that you can compare prices at various dealers. Find the list price of the car if your friend bought the car in:

a. Arizona, where the sales tax is 6.6%.

b. New York, where the sales tax is 8.25%.

c. A state where the sales tax is $r$.

e. Example: Solution (DOK 3)
Chase and his brother like to play basketball. About a month ago they decided to keep track of how many games they have each won. As of today, Chase has won 18 out of the 30 games against his brother.

a. How many games would Chase have to win in a row in order to have a 75% winning record?

b. How many games would Chase have to win in a row in order to have a 90% winning record?

c. Is Chase able to reach a 100% winning record? Explain why or why not.

d. Suppose that after reaching a winning record of 90% in part (b), Chase had a losing streak. How many games in a row would Chase have to lose in order to drop down to a winning record below 55% again?

f. Example: Solution (DOK 2)

Bernardo and Silvia play the following game. An integer between 0 and 999, inclusive, is selected and given to Bernardo. Whenever Bernardo receives a number, he doubles it and passes the result to Silvia. Whenever Silvia receives a number, she adds 50 to it and passes the result to Bernardo. The winner is the last person who produces a number less than 1000. What is the smallest initial number that results in a win for Bernardo?
g. Example: **Solution** (DOK 3)

a. Below is a quadrilateral \(ABCD\):

Show, by dividing \(ABCD\) into triangles, that the sum of the interior angles is 360°:

\[ m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360. \]

b. Below is a pentagon \(ABCDE\):

Show that the sum of the interior angles is 540°:

\[ m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) + m(\angle E) = 540. \]

c. Suppose \(P\) is a polygon with \(n \geq 3\) sides and assume that all interior angles of \(P\) measure less than 180 degrees. Show that the sum of the measures of the interior angles of \(P\) is

\[ (n - 2) \times 180 \text{ degrees}. \]

Check that this formula gives the correct value for equilateral triangles and squares.

h. Example: **Solution** (DOK 3)
When Marcus started high school, his grandmother opened a college savings account. On the first day of each school year she deposited money into the account: $1000 in his freshmen year, $600 in his sophomore year, $1100 in his junior year and $900 in his senior year. The account earns interest of $r\%$ at the end of each year. When Marcus starts college after four years, he gets the balance of the savings account plus an extra $500.

a. If $r$ is the annual interest rate of the bank account, the at the end of the year the balance in the account is multiplied by a growth factor of $x = 1 + r$. Find an expression for the total amount of money Marcus receives from his grandmother as a function of this annual growth factor $x$.

b. Suppose that altogether he receives $4400 from his grandmother. Use appropriate technology to find the interest rate that the bank account earned.

c. How much total interest did the bank account earn over the four years?

d. Suppose the bank account had been opened when Marcus started Kindergarten. Describe how the expression for the amount of money at the start of college would change. Give an example of what it might look like.
i. Example: Solution (DOK 3)

A common claim is that it is impossible to fold a single piece of paper in half more than 7 times (try it!).

Among other attempts, the challenge was taken up on an episode of the TV show Mythbusters, trying to avoid the physical restrictions by beginning with an exceptionally large sheet of paper. Watch the quick summary of their efforts below:

As you can see, the height of the folded sheet of paper increases dramatically as you continue folding. Assuming you started with a large enough sheet of paper, how many folds would it take for the stack of paper to reach the moon?

j. Example: Jim can paint a house in 12 hours. Alex can paint the same house in 8 hours. Write an equation that can be used to find the time in hours, t, it would take Jim and Alex to paint the house together.

k. Example: SPEEDING TICKETS

New York State wants to change its system for assigning speeding fines to drivers. The current system allows a judge to assign a fine that is within the ranges shown in Table 1.

<table>
<thead>
<tr>
<th>Miles per Hour over Speed Limit</th>
<th>Minimum Fine</th>
<th>Maximum Fine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 – 10</td>
<td>$45</td>
<td>$150</td>
</tr>
<tr>
<td>11 – 30</td>
<td>$90</td>
<td>$300</td>
</tr>
<tr>
<td>31 or more</td>
<td>$180</td>
<td>$600</td>
</tr>
</tbody>
</table>

Some people have complained that the New York speeding fine system is not fair. The New Drivers Association (NDA) is recommending a new speeding fine system. The NDA is studying the Massachusetts system because of claims that it is fairer than the New York system.
Table 2. Massachusetts Speeding Fines

<table>
<thead>
<tr>
<th>Miles per Hour over Speed Limit</th>
<th>Fine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 10</td>
<td>$100 flat charge</td>
</tr>
<tr>
<td>11 or more</td>
<td>$100 flat charge plus $10 for each additional mph above the first 10 mph</td>
</tr>
</tbody>
</table>

In this task, you will:

- Analyze the speeding fine systems for both New York and Massachusetts.
- Use data to propose a fairer speeding fine system for New York state.

Part B
Create an equation to calculate the Massachusetts speeding fine, $f$, based on the number of miles per hour, $m$, over the speed limit when $1 \leq m \leq 10$.

<table>
<thead>
<tr>
<th>Item</th>
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<td>#2</td>
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<td>A-CED</td>
<td>G</td>
<td>2</td>
<td>HAS.CED.A.1</td>
<td>2</td>
<td>See exemplar</td>
</tr>
</tbody>
</table>

For this item, a full-credit response (1 point) includes

- $f = 100$, and equivalent responses.

For this item, a no-credit response (0 points) includes none of the features of a full-credit response.

Part C
Create an equation to calculate the Massachusetts speeding fine, $f$, based on the number of miles per hour, $m$, over the speed limit when $m > 10$.

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<tr>
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<td>HAS.CED.A.1</td>
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<td>See exemplar</td>
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</table>

For this item, a full-credit response (1 point) includes
\[ f = 100 + 10(m - 10) \] or \[ f = 10(m - 10) + 100 \] or \[ f = 10m, \]
and equivalent responses.

For this item, a no-credit response (0 points) includes none of the features of a full-credit response.

\[ a. \] Example: The noise level at a music concert must be no more than 80 decibels (dB) at the edge of the property on which the concert is held. Melissa uses a decibel meter to test whether the noise level at the edge of the property is no more than 80 dB.

- Melissa is standing 10 feet away from the speakers and the noise level is 100 dB.
- The edge of the property is 70 feet away from the speakers.
- Every time the distance between the speakers and Melissa doubles, the noise level decreases by about 6 dB.

Rafael claims that the noise level at the edge of the property is no more than 80 dB since the edge of the property is over 4 times the distance from where Melissa is standing. Explain whether Rafael is or is not correct.

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<td>2</td>
<td>A-CED.1</td>
<td>N/A</td>
<td>See Below</td>
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</tbody>
</table>

Sample Top Score-response:
Rafael is not correct because the dB level does not decrease by at least \((6)(4)=24\). The decibel level decreases by 6 every time the distance is doubled starting from 10 feet. At 10 feet from the speakers, the volume is 100 dB. At 20 feet, it is 100-6=94 dB. At 40 feet, it is 94-6=88 dB. At 80 feet, it is 88-6=82 dB. Since the property line is 70 feet from the speakers, Rafael is wrong. The volume will be greater than 82 dB.

For full credit (2 points):
The response demonstrates a full and complete understanding of communicating reasoning. The response contains the following evidence:

- The student determines Rafael is incorrect.
  
  AND

- The student provides sufficient reasoning to support this conclusion.

For partial credit (1 point):
The response demonstrates a partial understanding of communicating reasoning. The response contains the following evidence:

- The student determines Rafael is incorrect but does not provide sufficient reasoning to support this conclusion.
b. Example: Tony is buying a used car. He will choose between two cars. The table below shows information about each car.

<table>
<thead>
<tr>
<th>Car</th>
<th>Cost</th>
<th>Miles Per Gallon (MPG)</th>
<th>Estimated Immediate Repairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car A</td>
<td>$3200</td>
<td>18</td>
<td>$700</td>
</tr>
<tr>
<td>Car B</td>
<td>$4700</td>
<td>24</td>
<td>$300</td>
</tr>
</tbody>
</table>

Tony wants to compare the total costs of buying and using these cars.
- Tony estimates he will drive at least 200 miles per month.
- The average cost of gasoline per gallon in his area is $3.70.
- Tony plans on owning the car for 4 years.

Calculate and explain which car will cost Tony the least to buy and use.

Sample Top-Score Response:
For Car A it will cost $3200 + 700 + (200 miles/month x 48 months x $3.70/gallon x 1 gallon/18 miles) = $5873.33

For Car B it will cost $4700 + 300 + (200 miles/month x 48 months x $3.70/gallon x 1 gallon/24 miles) = $6480

Tony will spend less money if he buys Car A.

For Full credit (2 points):
The response demonstrates a full and complete understanding of solving problems of this type. The response contains the following evidence:
- The student determines Car A will cost the least.
- The student provides sufficient reasoning to support this conclusion.

For partial credit (1 point):
The response demonstrates a partial understanding of solving problems of this type. The response contains the following evidence:
- The student determines Car A will cost the least; however, the student does not provide sufficient reasoning to support this conclusion.
OR

- The student selects Car B but provides reasoning to support this answer that contains a minor conceptual or computation error.

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. **(A-CED.A.2) (DOK 1, 2)**
   a. Example: Solution (DOK 2)
      
      A ball thrown vertically upward at a speed of $v$ ft/sec(384,513),(742,924) rises a distance $d$ feet in $t$ seconds, given by
      
      $$d = 6 + vt - 16t^2$$

      Write an equation whose solution is

      a. The time it takes a ball thrown at a speed of 88 ft/sec to rise 20 feet.
      b. The speed with which the ball must be thrown to rise 20 feet in 2 seconds.

   b. Example: Solution (DOK 2)

      It takes Clea 60 seconds to walk down an escalator when it is not operating, and only 24 seconds to walk down the escalator when it is operating. How many seconds does it take Clea to ride down the operating escalator when she just stands on it?

   c. Example: Solution (DOK 3)

      A tessellation of the plane is an arrangement of polygons which cover the plane without gaps or overlapping. For example, part of a tessellation with rectangles is pictured below:

      A tessellation is called *regular* if all polygons in the tessellation are congruent regular polygons and if any two polygons in the tessellation either do not meet, share a vertex only, or share one edge. The checkerboard pattern below is an example of a regular tessellation which can be continued indefinitely in all directions:
In this problem you will discover some very strong restrictions on possible tesselations of the plane, stemming from the fact that each interior angle of an $n$ sided regular polygon measures $\frac{180(n-2)}{n}$ degrees.

a. Suppose $P_n$ is a regular $n$ sided polygon and there is a tesselation of the plane by polygons congruent to $P_n$. Suppose that $m$ of these polygons meet at each vertex in the tesselation. Explain why

$$m \times \left( \frac{180(n - 2)}{n} \right) = 360.$$ 

b. Show that for any such tesselation, we must have $m \geq 3$ and, using part (a), that $n \leq 6$.

c. Using (a) and (b), find all possible pairs $(m, n)$ for a regular tesselation of the plane. For each possibility, draw a picture of the corresponding tesselation.

d. Example: Solution (DOK 2)
The distance between two points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) is given by
\[
\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.
\]

a. For each of the following equations, describe the set of solutions geometrically and sketch this solution set in \(x\)-\(y\)-\(z\) coordinates:
   i. \(x^2 + y^2 + z^2 = 1\).
   ii. \((x - 3)^2 + y^2 + z^2 = 4\).
   iii. \(x^2 + y^2 + \left(z - \frac{3}{2}\right)^2 = 1\).

b. Find the set of real numbers \((x, y, z)\) which are solutions both to \(x^2 + y^2 + z^2 = 1\) and to \((x - 3)^2 + y^2 + z^2 = 4\). Sketch this solution set in \(x\)-\(y\)-\(z\) coordinates.

c. Find the set of real numbers \((x, y, z)\) which are solutions to both \(x^2 + y^2 + z^2 = 1\) and \(x^2 + y^2 + \left(z - \frac{3}{2}\right)^2 = 1\). Sketch the solution set in \(x\)-\(y\)-\(z\) coordinates.

e. Example: Solution (DOK 3)

Sara's doctor tells her she needs between 400 and 800 milligrams of folate per day, with part coming from her diet and part coming from a multi-vitamin. Each multi-vitamin contains 50 mg of folate, and because of the inclusion of other vitamins and minerals, she can only take a maximum of 8 tablets per day.

   a. i. What are the possible combinations of \(n\), number of vitamin tablets taken, and \(a\), the amount of dietary folate, which will give Sarah exactly the minimum of 400 mg of folate each day? Express your answers in a table.

   ii. What are the possible combinations of vitamin tablets and dietary folate which give the maximum of 800 mg of folate each day?

   iii. Now use your tables from parts (i) and (ii) to express your answers as a system of three inequalities. Create a graph of \(a\) versus \(n\), and compare your graph to your tables from parts (i) and (ii).

b. Suppose instead of a multi-vitamin, Sara is given a powdered folate supplement she can add to water. She can drink any amount of the supplement she wants per day, as long as she does not exceed 400 mg per day. What are the possible combinations of folate she can ingest from her diet and from the powder? Graph your solution set. How does this graph compare to the graph from part (a)?
f. Example: Solution (DOK 2)

Below is a picture of a rectangle $ABCD$ with segment $MN$ drawn where $M$ is the midpoint of $BC$ and $N$ is the midpoint of $AD$:

![Diagram of rectangle ABCD with M and N as midpoints]

Suppose $ABCD$ is similar to $BMNA$. What is $\frac{|BC|}{|AB|}$?

g. Example: Solution (DOK 2)

Uranium 238 is a radioactive material with many applications in nuclear technology. It decays exponentially with a half-life of about 4.5 billion years.

a. Write an equation expressing how much Uranium 238 remains after $t$ years assuming the initial sample size is 100 grams.

b. For each graph below, decide whether or not the graph could represent the equation that you found in part (a):
   i. 

![Graph of Uranium decay]
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods. (A-CED.A.3) (DOK 1,2,3)

a. Example: Solution (DOK 3)
The only coins that Alexis has are dimes and quarters.

- Her coins have a total value of $5.80.
- She has a total of 40 coins.

Which of the following systems of equations can be used to find the number of dimes, \( d \), and the number of quarters, \( q \), Alexis has? Explain your choice.

a. 
\[
\begin{align*}
  d + q &= 5.80 \\
  40d + 40q &= 5.80
\end{align*}
\]

b. 
\[
\begin{align*}
  d + q &= 40 \\
  0.25d + 0.10q &= 5.80
\end{align*}
\]

c. 
\[
\begin{align*}
  d + q &= 5.80 \\
  0.10d + 0.25q &= 40
\end{align*}
\]

d. 
\[
\begin{align*}
  d + q &= 40 \\
  0.10d + 0.25q &= 5.80
\end{align*}
\]

b. Example: Solution (DOK 2)
In (a)–(d), (i) write a constraint equation, (ii) determine two solutions, and (iii) graph the equation and mark your solutions.

a. The relation between quantity of chicken and quantity of steak if chicken costs $1.29/lb and steak costs $3.49/lb, and you have $100 to spend on a barbecue.

b. The relation between the time spent walking and driving if you walk at 3 mph then hitch a ride in a car traveling at 75 mph, covering a total distance of 60 miles.

c. The relation between the volume of titanium and iron in a bicycle weighing 5 kg, if titanium has a density of 4.5g/cm\(^3\) and iron has a density of 7.87 g/cm\(^3\) (ignore other materials).

d. The relation between the time spent walking and the time spent canoeing on a 30 mile trip if you walk at 4 mph and canoe at 7 mph.

c. Example: Solution (DOK 2)
The coffee variety *Arabica* yields about 750 kg of coffee beans per hectare, while *Robusta* yields about 1200 kg per hectare (reference). Suppose that a plantation has \(a\) hectares of *Arabica* and \(r\) hectares of *Robusta*.

a. Write an equation relating \(a\) and \(r\) if the plantation yields 1,000,000 kg of coffee.

b. On August 14, 2003, the world market price of coffee was $1.42 per kg of *Arabica* and $0.73 per kg of *Robusta*. Write an equation relating \(a\) and \(r\) if the plantation produces coffee worth $1,000,000.

d. Example: Solution (DOK 3)

Fishing Adventures rents small fishing boats to tourists for day long fishing trips. Each boat can hold at most eight people. Additionally, each boat can only carry 1200 pounds of people and gear for safety reasons. Assume on average an adult weighs 150 pounds and a child weighs 75 pounds. Also assume each group will require 200 pounds of gear plus 10 pounds of gear per person.

a. Write an inequality that illustrates the weight limit for a group of adults and children on the fishing boat and a second inequality that represents the total number of passengers in the fishing boat. Graph the solution set to the inequalities.

b. Several groups of people wish to rent a boat. Group 1 has 4 adults and 2 children. Group 2 has 3 adults and 5 children. Group 3 has 8 adults. Which of the groups, if any, can safely rent a boat? What other combinations of adults and children are possible?

e. Example: Solution (DOK 2)

Bernardo and Silvia play the following game. An integer between 0 and 999, inclusive, is selected and given to Bernardo. Whenever Bernardo receives a number, he doubles it and passes the result to Silvia. Whenever Silvia receives a number, she adds 50 to it and passes the result to Bernardo. The winner is the last person who produces a number less than 1000. What is the smallest initial number that results in a win for Bernardo?

f. Example: Solution (DOK 2)
Sara's doctor tells her she needs between 400 and 800 milligrams of folate per day, with part coming from her diet and part coming from a multi-vitamin. Each multi-vitamin contains 50 mg of folate, and because of the inclusion of other vitamins and minerals, she can only take a maximum of 8 tablets per day.

a. i. What are the possible combinations of $n$, number of vitamin tablets taken, and $a$, the amount of dietary folate, which will give Sarah exactly the minimum of 400 mg of folate each day? Express your answers in a table.

ii. What are the possible combinations of vitamin tablets and dietary folate which give the maximum of 800 mg of folate each day?

iii. Now use your tables from parts (i) and (ii) to express your answers as a system of three inequalities. Create a graph of $a$ versus $n$, and compare your graph to your tables from parts (i) and (ii).

b. Suppose instead of a multi-vitamin, Sara is given a powdered folate supplement she can add to water. She can drink any amount of the supplement she wants per day, as long as she does not exceed 400 mg per day. What are the possible combinations of folate she can ingest from her diet and from the powder? Graph your solution set. How does this graph compare to the graph from part (a)?
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange Ohm’s law $V = IR$ to highlight resistance $R$. *(A-CED.A.4) (DOK 1)*

a. Example: Solution (DOK 2)

   Use inverse operations to solve the equations for the unknown variable, or for the designated variable if there is more than one. If there is more than one operation to “undo”, be sure to think carefully about the order in which you do them. For equations with multiple variables, it may help to first solve a version of the problem with numerical values substituted in.

   a. $5 = a - 3$
   b. $A - B = C$ (solve for $A$)
   c. $6 = -2x$
   d. $IR = V$ (solve for $R$)
   e. $\frac{x}{5} = 3$
   f. $W = \frac{4}{L}$ (solve for $A$)
   g. $7x + 3 = 10$
   h. $ax + c = R$ (solve for $x$)
   i. $13 = 15 - 4x$
   j. $2h = w - 3p$ (solve for $p$)
   k. $F = \frac{Gmm}{r^2}$ (solve for $G$)
b. Example: **Solution (DOK 2)**

In each of the equations below, rewrite the equation, solving for the indicated variable.

a. If $F$ denotes a temperature in degrees Fahrenheit and $C$ is the same temperature measured in degrees Celsius, then $F$ and $C$ are related by the equation

$$ F = \frac{9}{5} C + 32. $$

Rewrite this expression to solve for $C$ in terms of $F$.

b. The surface area $S$ of a sphere of radius $r$ is given by

$$ S = 4\pi r^2. $$

Solve for $r$ in terms of $S$.

c. The height $h$ of a diver over the water is modeled by the equation

$$ h = -5t^2 + 8t + 3 $$

where $h$ denotes the height of the diver over the water (in meters) and $t$ is time measured in seconds. Rewrite this equation, finding $t$ in terms of $h$.

d. A bacteria population $P$ is modeled by the equation

$$ P = P_0 10^{kt} $$

where time $t$ is measured in hours, $k$ is a positive constant, and $P_0$ is the bacteria population at the beginning of the experiment. Rewrite this equation to find $t$ in terms of $P$. 
Understand solving equations as a process of reasoning and explain the reasoning (A-REI.A)

Example: Determine values of $c$ and $d$ for which the equation

$$\sqrt{3x - 1} - \sqrt{cx + d} = 0$$

has no solution.

Enter a value for $c$ and a value for $d$.

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Example: Which statement is correct about the values of $x$ and $y$ in the following equation?

$$7x + xy = xy + 21$$

a. The equation is true for all ordered pairs $(x, y)$.
b. There are no $(x, y)$ pairs for which this equation is true.
c. For each value of $x$, there is one and only one value of $y$ that makes the equation true.
d. For each value of $y$, there is one and only one value of $x$ that makes the equation true.

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</table>

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. (A-REI.A.1) (DOK 1,2,3)

   a. Example: Solution (DOK 3)

   The following is a student solution to the inequality

   $\frac{5}{18} - \frac{x - 2}{9} \leq \frac{x - 4}{6}$
   $\frac{5}{18} - \frac{x - 2}{9} \leq \frac{x - 4}{6}$
   $\frac{5}{18} - \frac{2x - 2}{9} \leq \frac{3x - 4}{6}$
   $\frac{5}{18} - \frac{2x - 2}{9} \leq \frac{3x - 4}{6}$
   $\frac{5}{18} - \frac{(2x - 2)^2}{9} \leq \frac{3x - 4}{6}$
   $\frac{5}{18} - \frac{2x + 2}{3x - 4}$
   $7 - 2x \leq 3x - 4$
   $\frac{5}{18} - \frac{11}{x}$
   $x \leq \frac{11}{5}$

   a. There are two mathematical errors in this work. Identify at what step each mathematical error occurred and explain why it is mathematically incorrect.

   The first mathematical error occurred going from line ____ to line ____.

   Why it is incorrect:

   The second mathematical error occurred going from line ____ to line ____.
Why it is incorrect:

b. How would you help the student understand his mistakes?

c. Solve the inequality correctly.

d. Example: Solution (DOK 2)
   The product of two positive numbers is 9. The reciprocal of one of these
   numbers is 4 times the reciprocal of the other number. What is the sum
   of the two numbers?

c. Example: Solution (DOK 2)
   Working with complex numbers allows us to solve equations like
   \( z^2 = -1 \) which cannot be solved with real numbers. Here we will
   investigate complex numbers which arise as square roots of certain
   complex numbers.

   a. Find all complex square roots of -1, that is, find all numbers
      \( z = a + bi \) which satisfy \( z^2 = -1 \).

   b. Find all complex square roots of 1.

   c. Which complex numbers satisfy \( z^2 = i \)?

d. Example: Solution (DOK 3)
   In each of the following equations, the variables represent real
   numbers. Assuming each equation is true, what can you conclude
   about the values of the variables? Explain each step in your reasoning.

   a. \( 2z + 3 = 0 \)

   b. \( 7x = 0 \)

   c. \( 7(y - 5) = 0 \)

   d. \( ab = 0 \)

e. Example: Solution (DOK 3)
   The Zero Product Property states that if the product of two numbers is
   zero, then at least one of the numbers is zero. In symbols, where \( a \) and
   \( b \) represent numbers, if \( ab = 0 \), then \( a = 0 \) or \( b = 0 \). This steps below
   provide a proof of this property starting with the equation \( ab = 0 \).

   a. If \( a = 0 \), then the property is true. Explain.

   b. Assume that \( a \neq 0 \). Then \( a \) has a reciprocal. Explain.

   c. Since \( a \) has a reciprocal, we can multiply both sides of the equation
      by \( \frac{1}{a} \). What effect does this move have on the left side of the equation?
      On the right side of the equation?

   d. Explain why these steps prove the Zero Product Property.

f. Example: Solution (DOK 3)
The Zero Product Property (ZPP) states that if the product of two numbers is zero, then at least one of the numbers is zero. In symbols, if \( ab = 0 \), then \( a = 0 \) or \( b = 0 \). We can use this property when we solve equations where a product is 0. For each equation below, use the ZPP to find all solutions. Explain each step in your reasoning.

a. \( x(13 - 4x) = 0 \),
b. \( 7(y + 12) = 0 \),
c. \( (x - 19)(x + 3) = 0 \),
d. \( (y - 6)(3z - 4) = 0 \).

g. Example: Solution (DOK 3)

The Zero Product Property states that if the product of two numbers is zero, then at least one of the numbers is zero. In symbols, if \( ab = 0 \), then \( a = 0 \) or \( b = 0 \). Sometimes, we can take advantage of this property to help us find solutions to equations. Explain how the property can be used to find both solutions to each of the following equations, and explain each step in your reasoning.

a. \( (x - 1)(x - 3) = 0 \)
b. \( 2x(x - 1) + 3x - 3 = 0 \)
c. \( x + 4 = x(x + 4) \)
d. \( x^2 = 6x \)
e. \( x^2 + 10 = 7x \)
2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. (A-REI.A.2) (DOK 1,2)

a. Example: Solution (DOK 3)

   a. Solve the following two equations by isolating the radical on one side and squaring both sides:

   i. \(\sqrt{2x + 1} - 5 = -2\)
   ii. \(\sqrt{2x + 1} + 5 = 2\)

   Be sure to check your solutions.

b. If we raise both sides of an equation a power, we sometimes obtain an equation which has more solutions than the original one. (Sometimes the extra solutions are called extraneous solutions.) Which of the following equations result in extraneous solutions when you raise both sides to the indicated power? Explain.

   i. \(\sqrt{x} = 5\), square both sides
   ii. \(\sqrt{x} = -5\), square both sides
   iii. \(\sqrt[3]{x} = 5\), cube both sides
   iv. \(\sqrt[3]{x} = -5\), cube both sides

c. Create a square root equation similar to the one in part (a) that has an extraneous solution. Show the algebraic steps you would follow to look for a solution, and indicate where the extraneous solution arises.

b. Example: Solution (DOK 3)

   Chase and his brother like to play basketball. About a month ago they decided to keep track of how many games they have each won. As of today, Chase has won 18 out of the 30 games against his brother.

   a. How many games would Chase have to win in a row in order to have a 75% winning record?

   b. How many games would Chase have to win in a row in order to have a 90% winning record?

   c. Is Chase able to reach a 100% winning record? Explain why or why not.

   d. Suppose that after reaching a winning record of 90% in part (b), Chase had a losing streak. How many games in a row would Chase have to lose in order to drop down to a winning record below 55% again?

c. Example: Solution (DOK 3)
Alice and Briana each participate in a 5-kilometer race. Alice's distance covered, in kilometers, after $t$ minutes can be modeled by the equation $a(t) = \frac{t}{4}$. Briana's progress is modeled by the equation $b(t) = \sqrt{2t} - 1$.

b. Who gets to the finish line first? Explain.
c. At what time(s) during the race are Alice and Briana side by side? Explain.

d. Example: Solution (DOK 2)

Jamie and Ralph take a canoe trip up a river for 1 mile and then return. The current in the river is 1 mile per hour. The total trip time is 2 hours and 24 minutes. Assuming that they are paddling at a constant rate throughout the trip, find the speed that Jamie and Ralph are paddling.

e. Example: Solution (DOK 3)

Megan is working solving the equation

$$\frac{2}{x^2 - 1} - \frac{1}{x - 1} = \frac{1}{x + 1}.$$ 

She says

*If I clear the denominators I find that the only solution is $x = 1$ but when I substitute in $x = 1$ the equation does not make any sense.*

a. Is Megan's work correct?
b. Why does Megan's method produce an $x$ value that does not solve the equation?

f. Example: Select all equations that have at least one integer solution.

1. $\sqrt{4x} = 5$
2. $\sqrt{3x} = 75$
3. $\sqrt{x} = \frac{\sqrt{16}}{8}$
4. $\sqrt{x} = x - 12$
5. $\sqrt{10 - x} = x - 2$
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<td>2</td>
<td>HSA.REI.A.2</td>
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</table>

Solve equations and inequalities in one variable (A-REI.B)

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. (A-REI.B.3) (DOK 1)
   a. Example: Solution (DOK 3)

   The following is a student solution to the inequality
   \[
   \frac{5}{18} \left( \frac{x - 2}{9} \right) \leq \frac{x - 4}{6}.
   \]

   a. There are two mathematical errors in this work. Identify at what step each mathematical error occurred and explain why it is mathematically incorrect.

   The first mathematical error occurred going from line ____ to line ____.

   Why it is incorrect:

   The second mathematical error occurred going from line ____ to line ____

   b. Example: Example: Tony is buying a used car. He will choose between two cars. The table below shows information about each car.
Tony wants to compare the total costs of buying and using these cars.

- Tony estimates he will drive at least 200 miles per month.
- The average cost of gasoline per gallon in his area is $3.70.
- Tony plans on owning the car for 4 years.

Calculate and explain which car will cost Tony the least to buy and use.

<table>
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<tr>
<th>Item</th>
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<td>2A, 2C</td>
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<td>A.CED.1, A-REI.3</td>
<td>N/A</td>
<td>See Below</td>
</tr>
</tbody>
</table>

**Sample Top-Score Response:**

For Car A it will cost $3200 + 700 + (200 miles/month x 48 months x $3.70/gallon x 1 gallon/18 miles) = $5873.33

For Car B it will cost $4700 + 300 + (200 miles/month x 48 months x $3.70/gallon x 1 gallon/24 miles) = $6480

Tony will spend less money if he buys Car A.

**For Full credit (2 points):**

The response demonstrates a full and complete understanding of solving problems of this type. The response contains the following evidence:

- The student determines Car A will cost the least.

   AND

- The student provides sufficient reasoning to support this conclusion.

**For partial credit (1 point):**

The response demonstrates a partial understanding of solving problems of this type. The response contains the following evidence:

- The student determines Car A will cost the least; however, the student does not provide sufficient reasoning to support this conclusion.

   OR

- The student selects Car B but provides reasoning to support this answer that contains a minor conceptual or computation error.

4. Solve quadratic equations in one variable.
   a. Use the method of completing the square to transform any quadratic equation in \( x \) into an equation of the form \((x - p)^2 = q\) that has the same solutions. Derive the quadratic formula from
b. Solve quadratic equations by inspection (e.g., for \( x^2 = 49 \)), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as \( a \pm bi \) for real numbers \( a \) and \( b \). *(A-REI.B.4) (DOK 1,2,3)*

1. **Example:** *Solution* (DOK 3)

Suppose \( h(t) = -5t^2 + 10t + 3 \) is the height of a diver above the water (in meters), \( t \) seconds after the diver leaves the springboard.

a. How high above the water is the springboard? Explain how you know.

b. When does the diver hit the water?

c. At what time on the diver’s descent toward the water is the diver again at the same height as the springboard?

d. When does the diver reach the peak of the dive?

2. **Example:** *Solution* (DOK 3)

In "Building an explicit quadratic function" a particular quadratic function was rewritten by completing the square. The quadratic function used was \( q(x) = 2x^2 + 4x - 16 \) and this function was rewritten as

\[
q(x) = 2(x + 1)^2 - 18 = 2 \left( (x + 1)^2 - 9 \right).
\]

Some of the advantages to this form are that the \( x \)-coordinate of the vertex of the graph of \( q \) can be found more easily and the two roots of \( q \) can also be found readily. The right hand side of this equation can be seen as a horizontal translation by \(-1\), then squaring, then a vertical translation by \(-9\), and finally a multiplicative scaling by \(2\). The goal of this task is first to employ the same technique on a general quadratic function and then derive the quadratic formula. To assist in this process, we first rewrite the equation above:

\[
\frac{q(x)}{2} = (x + 1)^2 - 9.
\]

Let \( f \) be a quadratic function, so we have

\[
f(x) = ax^2 + bx + c.
\]

Here \( a, b, c \) are real numbers and we assume that \( a \) is non-zero.

a. Following the lead of our example problem, we begin by dividing out the leading coefficient:

Multiplying \( f(x) \) by \( \frac{1}{a} \) gives
Why do the expressions $f(x)$ and $\frac{f(x)}{a}$ have the same roots?

3. Example: Solution (DOK 3)
   The braking distance, in feet, of a car traveling at $v$ miles per hour is given by
   
   $$d = 2.2v + \frac{v^2}{20}.$$

   a. What is the braking distance, in feet, if the car is going 30 mph? 60 mph? 90 mph?
   b. Suppose that the car took 500 feet to brake. Use your computations in part (a) to make a prediction about how fast it was going when the brakes were applied.
   c. Use a graph of the distance equation to determine more precisely how fast it was going when the brakes were applied, and check your answer using the quadratic formula.

4. Example: Solution (DOK 3)
   Solve the quadratic equation
   
   $$x^2 = (2x - 9)^2$$

   using as many different methods as possible.

5. Example: Solution (DOK 3)
   Renee reasons as follows to solve the equation $x^2 + x + 1 = 0$.

   First I will rewrite this as a square plus some number.

   $$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

   Now I can subtract $\frac{3}{4}$ from both sides of the equation

   $$\left(x + \frac{1}{2}\right)^2 = -\frac{3}{4}.$$

   But I can't take the square root of a negative number so I can't solve this equation.

   a. Show how Renee might have continued to find the complex solutions of $x^2 + x + 1 = 0$.
   b. Apply Renee's reasoning to find the solutions to $x^2 + 4x + 6 = 0$.

6. Example: Solution (DOK 3)
Enrico has discovered a geometric technique for "completing the square" to find the solutions of quadratic equations. To solve the equation $x^2 + 6x + 4 = 0$, Enrico draws a square of dimension $x$ by $x$ and attaches 6 strips (3 of dimension 1 by $x$ and 3 of dimension $x$ by 1) to make the picture below:

![Image of the square and strips]

To complete the larger square, Enrico adds 9 squares of dimension 1 by 1. He has 4 of them in his initial expression so he needs five more as shown in the picture. So the picture represents the equation

$$(x + 3)^2 = (x^2 + 6x + 4) + 5$$

a. Explain how Enrico’s method helps find the roots of $x^2 + 6x + 4 = 0$.

b. Help Enrico draw a picture for, and then solve, the equation $x^2 - 6x + 4 = 0$.

7. Example: Solution (DOK 3)

Joanne wants to graph a quadratic function whose roots are $5 \pm 2i$, and says:

*I know the graph is a parabola, and the roots tell me that my function does not cross the x-axis, but I’m not sure where to go next -- how do I use this information to help with my graph?*

a. What can you deduce about the vertex of Joanne’s parabola?

b. With the information provided, can you graph Joanne’s function? Why?

8. Example: Solution (DOK 3)
The first three steps of three visual patterns are shown below.

Pattern A:

Pattern B:

Pattern C:

Here are functions that define how many tiles are in step \( n \), in no particular order:

\[
\begin{align*}
  f(n) &= 3n^2 \\
  g(n) &= n^2 + 4 \\
  h(n) &= n^2 - 1
\end{align*}
\]

a. Decide which function defines which pattern.

b. One of these patterns has a step with 432 tiles in it, one has a step with 195 tiles in it, and one of these patterns has a step with 404 tiles in it. Decide which is which, find the step that contains that number of tiles, and explain how you know.

c. Describe and justify the steps to solve this equation for \( x \).

\[ ax^2 + c = d \]

9. Example: **Solution** (DOK 3)
The first three steps of two visual patterns are shown below.

The number of tiles in step $n$ of Pattern D is defined by $d(n) = (n + 3)^2$. The number of tiles in step $n$ of Pattern E is defined by $e(n) = (n + 1)^2 - 2$.

a. For each pattern, decide whether there is a step with 167 tiles in it. If so, which step is it? If not, explain how you know.

b. For each pattern, decide whether there is a step with 169 tiles in it. If so, which step is it? If not, explain how you know.

c. Describe and justify the steps for solving the following equation for $x$:

$$a(x - h)^2 = k$$

10. Example: Solution (DOK 3)

The Zero Product Property states that if the product of two numbers is zero, then at least one of the numbers is zero. In symbols, if $ab = 0$, then $a = 0$ or $b = 0$. Sometimes, we can take advantage of this property to help us find solutions to equations. Explain how the property can be used to find both solutions to each of the following equations, and explain each step in your reasoning.

a. $(x - 1)(x - 3) = 0$

b. $2x(x - 1) + 3x - 3 = 0$

c. $x + 4 = x(x + 4)$

d. $x^2 = 6x$

e. $x^2 + 10 = 7x$

11. Example: Solve the following equation for $n$.

$18n^2 - 50 = 0$

Write out your solution. If there are two solutions, write out both solutions.
Solve systems of equations (A-REI.C)

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions. *(A-REI.C.5) (DOK 2,3)*
   a. Example: Solution (DOK 3)

   Lisa is working with the system of equations \( x + 2y = 7 \) and \( 2x - 5y = 5 \). She multiplies the first equation by 2 and then subtracts the second equation to find \( 9y = 9 \), telling her that \( y = 1 \). Lisa then finds that \( x = 5 \). Thinking about this procedure, Lisa wonders

   *There are lots of ways I could go about solving this problem. I could add 5 times the first equation and twice the second or I could multiply the first equation by -2 and add the second. I seem to find that there is only one solution to the two equations but I wonder if I will get the same solution if I use a different method?*

   a. What is the answer to Lisa's question? Explain.

   b. Does the answer to (a) change if we have a system of two equations in two unknowns with no solutions? What if there are infinitely many solutions?

6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. *(A-REI.C.6) (DOK 1,2)*
   a. Example: Solution (DOK 3)

   Nola was selling tickets at the high school dance. At the end of the evening, she picked up the cash box and noticed a dollar lying on the floor next to it. She said,

   *I wonder whether the dollar belongs inside the cash box or not.*

   The price of tickets for the dance was 1 ticket for $5 (for individuals) or 2 tickets for $8 (for couples). She looked inside the cash box and found $200 and ticket stubs for the 47 students in attendance. Does the dollar belong inside the cash box or not?

   b. Example: Solution (DOK 3)
In 1983 the composition of pennies in the United States was changed due, in part, to the rising cost of copper. Pennies minted after 1983 weigh 2.50 grams while the earlier copper pennies weigh 3.11 grams.

a. A roll of pennies contains 50 coins. If a particular roll of pennies weighs 138.42 grams, how many of the old heavier pennies and how many of the new lighter pennies does this roll contain?

b. Suppose you measure a roll of pennies on a scale which only reads the weight to the nearest gram. The scale says that the roll of pennies weighs 131 grams. What can you conclude about the pennies in this roll? What if the roll weighs 134 grams? Explain.

c. Some references list the weight of the newer pennies as 2.5 grams instead of 2.50 grams. If you only knew that the measurement of 2.5 grams was precise to the nearest tenth of a gram, what can you conclude about the pennies in the roll if the scale reads 138.4 grams?

c. Example: Solution (DOK 3)

In 1983 the composition of pennies in the United States was changed due, in part, to the rising cost of copper. Pennies minted after 1983 weigh 2.50 grams while the earlier copper pennies, from 1865 through 1982, weigh 3.11 grams. Pennies made between 1859 and 1864 had a different composition, with the same diameter, and weighed 4.67 grams.

a. A roll of pennies contains 50 coins. If a roll of pennies weighs 145 grams (to the nearest hundredth of a gram), how many pennies of each type does the roll contain?

b. If two rolls of pennies have the same weight, do they necessarily contain the same number of pennies of each of the three different weights? Explain.

c. What is the answer to part (b) if the pennies instead weigh 2.5, 3.1 and 4.7 grams respectively?

d. Example: Solution (DOK 2)

A type of pasta is made of a blend of quinoa and corn. How much of the pasta is quinoa and how much is corn? The pasta company is not disclosing the percentage of each ingredient in the blend. However, quinoa contains a higher percentage of protein than corn. At the company's website, they give some information about the protein content of different foods and their own pasta blend, see http://www.quinoa.net/199.html. Use the protein content of each ingredient to find out how much quinoa and how much corn is in one serving of the pasta.

e. Example: Solution (DOK 2)
Jerry was recently diagnosed with celiac disease. This means that he cannot eat any food containing gluten, in particular, he cannot eat pasta made from wheat. His mom has found a gluten free pasta that he likes a lot. It is called “Ancient Harvest Quinoa Pasta”. How much of the pasta is actually quinoa?

f. Example: Solution (DOK 2)

The sums of three whole numbers taken in pairs are 12, 17, and 19. What is the middle number?

g. Example: Solution (DOK 3)

Without graphing, construct a system of two linear equations where \((-2, 3)\) is a solution to the first equation but not to the second equation, and where \((5, -2)\) is a solution to your system.

After you have created your system of equations, graph your system. Explain how your graph shows that your system satisfies the required conditions.

h. Example: Solution (DOK 3)
Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$. (A-REI.C.7) (DOK 1,2)

a. Example: Solution (DOK 3)
   Sketch the circle with equation $x^2 + y^2 = 1$ and the line with equation $y = 2x - 1$ on the same pair of axes.

   a. There is one solution to the pair of equations

      $\begin{align*}
      x^2 + y^2 &= 1 \\
      2x - 1 &= y
      \end{align*}$

      that is clearly identifiable from the sketch. What is it? Verify that it is a solution.

   b. Find all the solutions to this pair of equations.

b. Example: Solution (DOK 2)
The figure shows graphs of a linear and a quadratic function.

a. What are the coordinates of the point Q?

b. What are the coordinates of the point P?

c. Example: Solution (DOK 3)

This task will investigate the intersection points of the circle $C$ of radius 1 centered at $(0, 0)$ and different lines passing through the point $(0, 1)$. Recall that $C$ is the set of solutions to the equation $x^2 + y^2 = 1$. Also recall that a Pythagorean Triple is a set of three positive whole numbers $a, b, c$ so that $a^2 + b^2 = c^2$.

a. Find the two points of intersection of the line $y = -2x + 1$ and the circle of radius 1 with center $(0, 0)$.

b. Find the two points of intersection of the line $y = 4x + 1$ and the circle of radius 1 with center $(0, 0)$.

c. Suppose $m$ is a (non-zero) rational number and we intersect the line defined by $y = mx + 1$ with the circle $C$. Explain why $(0, 1)$ is one of the solutions and the other solution has rational numbers for both coordinates.

d. Parts (a), (b), and (c) produce a point on $C$ different from $(1, 0)$. Explain how this point can be used to produce a Pythagorean triple.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable. (A-REI.C.8) (DOK 1)

9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension $3 \times 3$ or greater). (A-REI.C.9) (DOK 1,2)
Represent and solve equations and inequalities graphically (A-REI.D)

8. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). *(A-REI.D.10) [DOK 1]*
   a. Example: Solution [DOK 3]

Lauren keeps records of the distances she travels in a taxi and what she pays:

<table>
<thead>
<tr>
<th>Distance, $d$, in miles</th>
<th>Fare, $F$, in dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8.25</td>
</tr>
<tr>
<td>5</td>
<td>12.75</td>
</tr>
<tr>
<td>11</td>
<td>26.25</td>
</tr>
</tbody>
</table>

a. If you graph the ordered pairs $(d, F)$ from the table, they lie on a line. How can you tell this without graphing them?

b. Show that the linear function in part (a) has equation $F = 2.25d + 1.5$.

c. What do the 2.25 and the 1.5 in the equation represent in terms of taxi rides?

b. Example: Solution [DOK 2]

The figure shows graphs of a linear and a quadratic function.

a. What are the coordinates of the point Q?

b. What are the coordinates of the point P?

c. Example: Solution [DOK 3]
Consider three points in the plane, $P = (-4, 0), Q = (-1, 12)$ and $R = (4, 32)$.

a. Find the equation of the line through $P$ and $Q$.

b. Use your equation in (a) to show that $R$ is on the same line as $P$ and $Q$.

c. Show that $P, Q$ and $R$ are on the graph of the equation $y = x^3 + x^2 - 12x$.

d. Is it possible for $P, Q$ and $R$ to all lie on a parabola of the form $y = ax^2 + bx + c$?

d. Example: The graph of an exponential function $f$ passes through $(0, 1)$ and $(2, 4)$, as shown.

What is the value of $f(6)$?

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<td>2</td>
<td>HSA.REI.D.10</td>
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<td></td>
</tr>
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</table>

9. Explain why the $x$-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

(A-REI.D.11) (DOK 1,2,3)

a. Example: Solution (DOK 3)
Solve the quadratic equation

\[ x^2 = (2x - 9)^2 \]

using as many different methods as possible.

b. Example: Solution (DOK 3)

The population of a country is initially 2 million people and is increasing at 4% per year. The country’s annual food supply is initially adequate for 4 million people and is increasing at a constant rate adequate for an additional 0.5 million people per year.

a. Based on these assumptions, in approximately what year will this country first experience shortages of food?

b. If the country doubled its initial food supply and maintained a constant rate of increase in the supply adequate for an additional 0.5 million people per year, would shortages still occur? In approximately which year?

c. If the country doubled the rate at which its food supply increases, in addition to doubling its initial food supply, would shortages still occur?

c. Example: Solution (DOK 3)

When Marcus started high school, his grandmother opened a college savings account. On the first day of each school year she deposited money into the account: $1000 in his freshmen year, $600 in his sophomore year, $1100 in his junior year and $900 in his senior year. The account earns interest of \( r \)% at the end of each year. When Marcus starts college after four years, he gets the balance of the savings account plus an extra $500.

a. If \( r \) is the annual interest rate of the bank account, the at the end of the year the balance in the account is multiplied by a growth factor of \( x = 1 + r \). Find an expression for the total amount of money Marcus receives from his grandmother as a function of this annual growth factor \( x \).

b. Suppose that altogether he receives $4400 from his grandmother. Use appropriate technology to find the interest rate that the bank account earned.

c. How much total interest did the bank account earn over the four years?

d. Suppose the bank account had been opened when Marcus started Kindergarten. Describe how the expression for the amount of money at the start of college would change. Give an example of what it might look like.
d. Example: Solution (DOK 2)

A certain number of Xenon gas molecules are placed in a container at room temperature. If $V$ is the volume of the container and $P(V)$ is the pressure exerted on the container by the Xenon molecules, a model predicts that

$$P(V) = \frac{40}{2V - 1} - \left(\frac{4}{V}\right)^2$$

for all $V > \frac{1}{2}$. Here the units for volume are liters and the units for pressure are atmospheres.

a. Sketch a graph of $P$.

b. Using the graph, approximate the volume for which the pressure is 10 atmospheres.

e. Example: This graph shows linear equations $y = f(x)$ and $y = g(x)$.

![Graph of linear equations]

Enter the solution to the equation $f(x) - g(x) = 0$

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<td>HSA.REI.D.11</td>
<td>N/A</td>
<td>_4.2 to _4</td>
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10. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes. (A-REI.D.12) (DOK 1,2)

a. Example: Solution (DOK 3)
Fishing Adventures rents small fishing boats to tourists for day long fishing trips. Each boat can hold at most eight people. Additionally, each boat can only carry 1200 pounds of people and gear for safety reasons. Assume on average an adult weighs 150 pounds and a child weighs 75 pounds. Also assume each group will require 200 pounds of gear plus 10 pounds of gear per person.

a. Write an inequality that illustrates the weight limit for a group of adults and children on the fishing boat and a second inequality that represents the total number of passengers in the fishing boat. Graph the solution set to the inequalities.

b. Several groups of people wish to rent a boat. Group 1 has 4 adults and 2 children. Group 2 has 3 adults and 5 children. Group 3 has 8 adults. Which of the groups, if any, can safely rent a boat? What other combinations of adults and children are possible?

b. Example: Solution (DOK 3)

Given below are the graphs of two lines, \( y = -0.5x + 5 \) and \( y = -1.25x + 8 \), and several regions and points are shown. Note that \( C \) is the region that appears completely white in the graph.

a. For each region and each point, write a system of equations or inequalities, using the given two lines, that has the region or point as its solution set and explain the choice of \( \leq, \geq, \) or \( = \) in each case. (You may assume that the line is part of each region.)

b. The coordinates of a point within a region have to satisfy the corresponding system of inequalities. Verify this by picking a specific point in each region and showing that the coordinates of this point satisfy the corresponding system of inequalities for that region.

c. In the previous part, we checked that specific coordinate points satisfied our inequalities for each region. Without picking any specific numbers, use the same idea to explain how you know that all points in the 3rd quadrant must satisfy the inequalities for region A.
Mathematics | High School—Functions

Functions describe situations where one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car’s speed in miles per hour, \( v \); the rule \( T(v) = \frac{100}{v} \) expresses this relationship algebraically and defines a function whose name is \( T \).

The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context.

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city;" by an algebraic expression like \( f(x) = a + bx \); or by a recursive rule. The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function can throw light on the function's properties.

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships.

A graphing utility or a computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions.

Connections to Expressions, Equations, Modeling, and Coordinates.
Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the same value for the same input lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.
Functions Overview

Interpreting Functions
- Understand the concept of a function and use function notation
- Interpret functions that arise in applications in terms of the context
- Analyze functions using different representations

Building Functions
- Build a function that models a relationship between two quantities
- Build new functions from existing functions

Linear, Quadratic, and Exponential Models
- Construct and compare linear, quadratic, and exponential models and solve problems
- Interpret expressions for functions in terms of the situation they model

Trigonometric Functions
- Extend the domain of trigonometric functions using the unit circle
- Model periodic phenomena with trigonometric functions
- Prove and apply trigonometric identities

Mathematical Practices
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
Interpreting Functions

Understand the concept of a function and use function notation (F-IF.A)

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If \( f \) is a function and \( x \) is an element of its domain, then \( f(x) \) denotes the output of \( f \) corresponding to the input \( x \). The graph of \( f \) is the graph of the equation \( y = f(x) \). (F-IF.A.1) [DOK 1]

   a. Example: Solution (DOK 3)
      
      a. Suppose \( P_1 = (0, 5) \) and \( P_2 = (3, -3) \). Sketch \( P_1 \) and \( P_2 \). For which real numbers \( m \) and \( b \) does the graph of a linear function described by the equation \( f(x) = mx + b \) contain \( P_1 \)? Explain. Do any of these graphs also contain \( P_2 \)? Explain.

      b. Suppose \( P_1 = (0, 5) \) and \( P_2 = (0, 7) \). Sketch \( P_1 \) and \( P_2 \). Are there real numbers \( m \) and \( b \) for which the graph of a linear function described by the equation \( f(x) = mx + b \) contains \( P_1 \) and \( P_2 \)? Explain.

      c. Now suppose \( P_1 = (c, d) \) and \( P_2 = (g, h) \) and \( c \) is not equal to \( g \). Show that there is only one real number \( m \) and only one real number \( b \) for which the graph of \( f(x) = mx + b \) contains the points \( P_1 \) and \( P_2 \).

   b. Example: Solution (DOK 3)
      
      A parking lot charges $0.50 for each half hour or fraction thereof, up to a daily maximum of $10.00. Let \( C(t) \) be the cost in dollars of parking for \( t \) minutes.

      a. Complete the table below.

      | \( t \) (minutes) | \( C(t) \) (dollars) |
      |-----------------|-------------------|
      | 0               |                   |
      | 15              |                   |
      | 20              |                   |
      | 35              |                   |
      | 75              |                   |
      | 125             |                   |

      b. Sketch a graph of \( C \) for \( 0 \leq t \leq 480 \).

      c. Is \( C \) a function of \( t \)? Explain your reasoning.

      d. Is \( t \) a function of \( C \)? Explain your reasoning.

   c. Example: Solution (DOK 3)
a. Let $F$ assign to each student in your math class his/her biological father. Explain why $F$ is a function.

b. Describe conditions on the class that would have to be true in order for $F$ to have an inverse.

c. In a case from part (b) in which $F$ does not have an inverse, can you modify the domain so that it does?

d. Example: Solution (DOK 3)

   a. Explain why the equation $y = x^2$ represents $y$ as a function of a real variable $x$.

   b. For the relation considered in part (a), is $x$ a function of $y$? Explain.

   c. Give a context in which the equation $y = x^2$ does represent $x$ as a function of $y$.

e. Example: Solution (DOK 3)

Katy is told that the cost of producing $x$ DVDs is given by $C(x) = 1.25x + 2500$. She is then asked to find an equation for $\frac{C(x)}{x}$, the average cost per DVD of producing $x$ DVDs.

She begins her work:

$$\frac{C(x)}{x} = \frac{1.25x + 2500}{x}$$

and finishes by simplifying both sides to get:

$$C = 1.25 + \frac{2500}{x}$$

Is Katy's answer correct? Explain.

f. Example: Solution (DOK 3)
A certain business keeps a database of information about its customers.

a. Let $C$ be the rule which assigns to each customer shown in the table his or her home phone number. Is $C$ a function? Explain your reasoning.

<table>
<thead>
<tr>
<th>Customer Name</th>
<th>Home Phone Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heather Baker</td>
<td>3105100091</td>
</tr>
<tr>
<td>Mike London</td>
<td>3105200256</td>
</tr>
<tr>
<td>Sue Green</td>
<td>3234132598</td>
</tr>
<tr>
<td>Bruce Swift</td>
<td>3234132598</td>
</tr>
<tr>
<td>Michelle Metz</td>
<td>2138061124</td>
</tr>
</tbody>
</table>

b. Let $P$ be the rule which assigns to each phone number in the table above, the customer name(s) associated with it. Is $P$ a function? Explain your reasoning.

c. Explain why a business would want to use a person’s social security number as a way to identify a particular customer instead of their phone number.

g. **Example:** Solution (DOK 2)

Suppose $f$ is a function.

a. If $10 = f(-4)$, give the coordinates of a point on the graph of $f$.

b. If 6 is a solution of the equation $f(u) = 1$, give a point on the graph of $f$.

h. **Example:** Solution (DOK 2)

For the functions in (a)-(f),

a. $y = \frac{2}{x - 3}$

b. $y = \sqrt{x - 5} + 1$

c. $y = 4 - (x - 3)^2$

d. $y = \frac{7}{4 - (x - 3)^2}$

e. $y = 4 - (x - 3)^{1/2}$

f. $y = \frac{7}{4 - (x - 3)^{1/2}}$
i. Example: Solution (DOK 2)

For the function

\[ f(x) = \frac{2}{x - 3} \]

a. Evaluate \( f(11) \), writing out every step. Write the output in decimal form.

b. Evaluate \( f(3) \), writing out every step. You will run into some trouble—describe it.

c. When you evaluate this function at an input, what operations are performed, and in what order? List them. What restrictions does each operation place on the domain of the function?

d. Give a possible domain for \( f \).

j. Example: The graphs of \( y = g(x) \) and \( y = f(x) \) are shown. Add a point that will satisfy each given condition.

- A point on the graph of \( g \) where \( x = 0 \)
- A point on the graph of \( g \) where \( f(x) > g(x) \)
- A point on the graph of \( f \) where \( f(x) = 0 \)
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. (F-IF.A.2) (DOK 1,2)
   a. Example: Solution (DOK 3)

   Given a function \( f \), is the statement
   \[
   f(x + h) = f(x) + f(h)
   \]
   true for any two numbers \( x \) and \( h \)? If so, prove it. If not, find a function for which the statement is true and a function for which the statement is false.

   b. Example: Solution (DOK 3)

   You put a yam in the oven. After 45 minutes, you take it out. Let \( f(t) \) be the temperature of the yam \( t \) minutes after you placed it in the oven.

   In (a)-(d), explain the meaning of the statement in everyday language.

   a. \( f(0) = 65 \)
   b. \( f(5) < f(10) \)
   c. \( f(40) = f(45) \)
   d. \( f(45) > f(60) \)

   c. Example: Solution (DOK 2)
Imagine Scott stood at zero on a life-sized number line. His friend flipped a coin 50 times. When the coin came up heads, he moved one unit to the right. When the coin came up tails, he moved one unit to the left. After each flip of the coin, Scott’s friend recorded his position on the number line.

a. Let \( f \) assign to the whole number \( n \), when \( 1 \leq n \leq 50 \), Scott’s position on the number line after the \( n \text{th} \) coin flip. Explain why \( f \) is a function.

b. Write a sentence explaining what \( f(5) = 5 \) means in everyday language.

c. Before Scott began the random walk, he asked his friend to calculate the probability that \( f(3) = 0 \). What should his friend respond?

d. **Example: Solution** (DOK 3)

Let \( f(t) \) be the number of people, in millions, who own cell phones \( t \) years after 1990. Explain the meaning of the following statements.

a. \( f(10) = 100.3 \)

b. \( f(a) = 20 \)

c. \( f(20) = b \)

d. \( n = f(t) \)

e. **Example: Solution** (DOK 3)

Imagine Scott stood at zero on a life-sized number line. His friend flipped a coin 50 times. When the coin came up heads, he moved one unit to the right. When the coin came up tails, he moved one unit to the left. After each flip of the coin, Scott’s friend recorded his position on the number line. Let \( f \) assign to the whole number \( n \), when \( 1 \leq n \leq 50 \), Scott’s position on the number line after the \( n \text{th} \) coin flip.

a. If \( f(6) = 6 \) what can you conclude about the outcomes of the first 6 coin tosses? Explain. What if \( f(6) = -4 \)?

b. Is it possible that \( f(7) = 2 \)? Explain.

c. Find all integers \( m \) so that the probability that \( f(50) = m \) is zero.

f. **Example:** Consider this function given in recursive form.

\[
\begin{align*}
  f(1) & = -3 \\
  f(n) & = 3f(n-1); n \geq 2
\end{align*}
\]

Select the equivalent explicit function for \( n \geq 1 \).
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$. (F-IF.A.3) (DOK 1)
   a. Example: Solution (DOK 2)

   In a video game called Snake, a player moves a snake through a square region in the plane, trying to eat the white pellets that appear.

   ![Snake game](image)

   If we imagine the playing field as a 32-by-32 grid of pixels, then the snake starts as a 4-by-1 rectangle of pixels, and grows in length as it eats the pellets:

   - After the first pellet, it grows in length by one pixel.
   - After the second pellet, it further grows in length by two pixels.
   - After the third pellet, it further grows in length by three pixels.
   - and so on, with the $n$-th pellet increasing its length by $n$ pixels.
Let $L(n)$ denote the length of the snake after eating $n$ pellets. For example, $L(3) = 10$.

a. How long is the snake after eating 4 pellets? After 5 pellets? After 6 pellets?

b. Find a recursive description of the function $L(n)$.

c. Find a non-recursive expression for $L(100)$, and evaluate that expression to compute $L(100)$.

d. What is the largest number of pellets a snake could eat before he could no longer fit in the playing field? That is, how long is a perfect game of snake?

Interpret functions that arise in applications in terms of the context (F-IF.B)

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* (F-IF.B.4) (DOK 1,2)

a. Example: Solution (DOK 3)

Mike likes to canoe. He can paddle 150 feet per minute. He is planning a river trip that will take him to a destination about 30,000 feet upstream (that is, against the current). The speed of the current will work against the speed that he can paddle.

a. Let $s$ be the speed of the current in feet per minute. Write an expression for $r(s)$, the speed at which Mike is moving relative to the river bank, in terms of $s$.

b. Mike wants to know how long it will take him to travel the 30,000 feet upstream. Write an expression for $T(s)$, the time in minutes it will take, in terms of $s$.

c. What is the vertical intercept of $T$? What does this point represent in terms of Mike’s canoe trip?

d. At what value of $s$ does the graph have a vertical asymptote? Explain why this makes sense in the situation.

e. For what values of $s$ does $T(s)$ make sense in the context of the problem?

b. Example: Solution (DOK 3)
John makes DVDs of his friend's shows. He has realized that, because of his fixed costs, his average cost per DVD depends on the number of DVDs he produces. The cost of producing \( x \) DVDs is given by

\[ C(x) = 2500 + 1.25x. \]

a. John wants to figure out how much to charge his friend for the DVDs. He's not trying to make any money on the venture, but he wants to cover his costs. Suppose John made 100 DVDs. What is the cost of producing this many DVDs? How much is this per DVD?

b. John is hoping to make many more than 100 DVDs for his friends. Complete the table showing his costs at different levels of production.

<table>
<thead>
<tr>
<th># of DVDs</th>
<th>0</th>
<th>10</th>
<th>100</th>
<th>1,000</th>
<th>10,000</th>
<th>100,000</th>
<th>1,000,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Cost</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost per DVD</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Explain why the average cost per DVD levels off.

d. Find an equation for the average cost per DVD of producing \( x \) DVDs.

e. Find the domain of the average cost function.

f. Using the data points from your table above, sketch the graph of the average cost function. How does the graph reflect that the average cost levels off?
c. Example: Solution (DOK 3)

Mike likes to canoe. He can paddle 150 feet per minute. He is planning a river trip that will take him to a destination about 30,000 feet upstream (that is, against the current). The speed of the current will work against the speed that he can paddle.

a. Mike guesses that the current is flowing at a speed of 50 feet per minute. Assuming this is correct, how long will it take for Mike to reach his destination?

b. Mike does not really know the speed of the current. Make a table showing the time it will take him to reach his destination for different speeds:

<table>
<thead>
<tr>
<th>Speed of Current (feet per minute)</th>
<th>Mike's Speed (feet per minute)</th>
<th>Time for Mike to travel 30,000 feet (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>140</td>
<td></td>
<td></td>
</tr>
<tr>
<td>149</td>
<td></td>
<td></td>
</tr>
<tr>
<td>s</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. The time $T$ taken by the trip, in minutes, as a function of the speed of the current is $s$ feet/minute. Write an equation expressing $T$ in terms of $s$. Explain why $s = 150$ does not make sense for this function, both in terms of the canoe trip and in terms of the equation.

d. Sketch a graph of the equation in part (c). Explain why it makes sense that the graph has a vertical asymptote at $s = 150$. 
d. Example: Solution (DOK 3)
A wheel of radius 0.2 meters begins to move along a flat surface so that the center of the wheel moves forward at a constant speed of 2.4 meters per second. At the moment the wheel begins to turn, a marked point \( P \) on the wheel is touching the flat surface.

\[
\text{\begin{tikzpicture}
\draw (0,0) circle (0.2);
\draw (-1.2,0) -- (1.2,0);
\draw (0,0) -- (0,0.2);
\node at (0,-0.2) {P};
\node at (0,0.3) {0.2 m};
\node at (1.2,0) {2.4 m/s};
\end{tikzpicture}}
\]

a. Write an algebraic expression for the function \( y \) that gives the height (in meters) of the point \( P \), measured from the flat surface, as a function of \( t \), the number of seconds after the wheel begins moving.

b. Sketch a graph of the function \( y \) for \( t > 0 \). What do you notice about the graph? Explain your observations in terms of the real-world context given in this problem.

c. We define the horizontal position of the point \( P \) to be the number of meters the point has traveled forward from its starting position, disregarding any vertical movement the point has made. Write an algebraic expression for the function \( x \) that gives the horizontal position (in meters) of the point \( P \) as a function of \( t \), the number of seconds after the wheel begins moving.

d. Sketch a graph of the function \( x \) for \( t > 0 \). Is there a time when the point \( P \) is moving backwards? Use your graph to justify your answer.
e. Example: Solution (DOK 3)

An epidemic of influenza spreads through a city. The figure below is the graph of \( I = f(w) \), where \( I \) is the number of individuals (in thousands) infected \( w \) weeks after the epidemic begins.

```
\[ I \]
```

```
f(w)
```

```
w
```

a. Estimate \( f(2) \) and explain its meaning in terms of the epidemic.

b. Approximately how many people were infected at the height of the epidemic? When did that occur? Write your answer in the form \( f(a) = b \).

c. For approximately which \( w \) is \( f(w) = 4.5 \); explain what the estimates mean in terms of the epidemic.

d. An equation for the function used to plot the image above is \( f(w) = 6w(1.3)^{-w} \). Use the graph to estimate the solution of the inequality \( 6w(1.3)^{-w} \geq 6 \). Explain what the solution means in terms of the epidemic.

(Task from Functions Modeling Change: A Preparation for Calculus, Connolly et al., Wiley 2010.)

f. Example: Solution (DOK 2)

The figure shows the graph of \( T \), the temperature (in degrees Fahrenheit) over one particular 20-hour period in Santa Elena as a function of time \( t \).

```
y = T(t)
```

```
8 16 24
```

a. Estimate \( T(14) \).

b. If \( t = 0 \) corresponds to midnight, interpret what we mean by \( T(14) \) in words.

c. Estimate the highest temperature during this period from the graph.

d. When was the temperature decreasing?

e. If Anya wants to go for a two-hour hike and return before the temperature gets over 80 degrees, when should she leave?
Example: Solution (DOK 3)

Given below are three graphs that show solar radiation, $S$, in watts per square meter, as a function of time, $t$, in hours since midnight. We can think about this quantity as the maximum amount of power that a solar panel can absorb, which tells us how intense the sunshine is at any given time. Match each graph to the corresponding description of the weather during the day.

a. It was a beautifully sunny day from sunrise to sunset – not a cloud in the sky.

b. The day started off foggy but eventually the fog lifted and it was sunny the rest of the day.

c. It was a pretty gloomy day. The morning fog never really lifted.

1.

![Graph of solar radiation vs. time]

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SOLAR RADIATION - W/M^2 (13451)
All three graphs show solar radiation measured in Santa Rosa, a city in northern California. What other information can you get from the graph?

h. Example: Solution (DOK 3)
Each of the following graphs tells a story about some aspect of the weather: temperature (in degrees Fahrenheit), solar radiation (in watts per square meters), and cumulative rainfall (in inches) measured by sensors in Santa Rosa, CA in February 2012. Note that the vertical gridlines represent the start of the day whose date is given.

a. Give a verbal description of the function represented in each graph. What does each function tell you about the weather in Santa Rosa?

b. Tell a more detailed story using information across several graphs. What are the connections between the graphs?

- Example: Solution (DOK 3)
An important example of a model often used in biology or ecology to model population growth is called the logistic growth model. The general form of the logistic equation is

\[ P(t) = \frac{K P_0 e^{rt}}{K + P_0 (e^{rt} - 1)}. \]

In this equation \( t \) represents time, with \( t = 0 \) corresponding to when the population in question is first measured; \( K, P_0 \) and \( r \) are all real numbers with \( K \) being called the "carrying capacity" while \( r \) is a growth rate and is normally a positive number.

a. Explain why the value \( P_0 \) represents the population when it is first measured.

b. Explain why, as time elapses, the population stabilizes, approaching the value \( K \).

c. Explain how the behavior of \( P \) changes if the growth rate \( r \) is increased or decreased.

d. Below is the graph of a particular logistic function \( P \), showing the growth of a bacteria population. Using the graph, identify \( P_0 \) and \( K \).

![Graph of logistic function](image)

<table>
<thead>
<tr>
<th>Population in units of millions</th>
<th>Time in hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

\[ y = P(t) \]

\[ P_0 \]

\[ K \]

e. Using the values of \( P_0 \) and \( K \) from the previous part, sketch the graph of the logistic function \( Q \) given by

\[ Q(t) = \frac{K P_0 e^{2rt}}{K + P_0 (e^{2rt} - 1)}. \]

Note that \( Q \) is the same as \( P \) except that the growth rate \( r \) has been doubled.

j. Example: Solution (DOK 3)
The dots in the graph below show the approximate United States Population measured each decade starting in 1790 up through 1940:

The curve above, modeling the United States population, is the graph of the function \( P \) given by the rule

\[
P(t) = \frac{(3,900,000 \times 200,000,000) e^{0.31t}}{200,000,000 + 3,900,000 (e^{0.31t} - 1)}.
\]

a. According to this model for the U.S. population, what was the population in the year 1790?

b. According to this model for the U.S. population, when did the population first reach 100,000,000? Explain. How much does this differ from an estimate that the U.S. population first reached 100,000,000 in 1915?

c. According to this model, what should the population of the U.S. be in the year 2010? How much does this differ from the Wikipedia estimate of 309,000,000?

d. For larger values of \( t \), such as \( t = 50 \), what does this model predict for the U.S. population? Why?

k. Example: Solution (DOK 3)
Suppose Brett and Andre each throw a baseball into the air. The height of Brett's baseball is given by

\[ h(t) = -16t^2 + 79t + 6, \]

where \( h \) is in feet and \( t \) is in seconds. The height of Andre's baseball is given by the graph below:

Brett claims that his baseball went higher than Andre's, and Andre says that his baseball went higher.

a. Who is right?

b. How long is each baseball airborne?

c. Construct a graph of the height of Brett's throw as a function of time on the same set of axes as the graph of Andre's throw (if not done already), and explain how this can confirm your claims to parts (a) and (b).

1. **Example: Solution** (DOK 2)

A model airplane pilot is practicing flying her airplane in a big loop for an upcoming competition. At time \( t = 0 \) her airplane is at the bottom of the loop 100 feet above the ground. The loop is a supposed to be a perfect circle, at its highest point the airplane is 400 feet above the ground, and it takes her 60 seconds to complete one loop. The airplane completes two loops and it is supposed to fly at a constant speed the entire time to score well in the competition.

a. The diagram below is a representation of the loop of the airplane. Draw in points that show where the airplane is every 15 seconds, starting at \( t = 0 \). Assume that the plane is going around the circle in the counterclockwise direction.
b. Make a table of values of the height of the airplane in feet above the ground as a function of time in seconds for the two loops.

c. Draw a graph of the height of the airplane during the two minute time interval on the axes below.

Example: Solution (DOK 3)

The table below shows historical estimates for the population of London.

<table>
<thead>
<tr>
<th>Year</th>
<th>1801</th>
<th>1821</th>
<th>1841</th>
<th>1861</th>
<th>1881</th>
<th>1901</th>
<th>1921</th>
<th>1941</th>
<th>1961</th>
</tr>
</thead>
<tbody>
<tr>
<td>London population</td>
<td>1,100,000</td>
<td>1,600,000</td>
<td>2,200,000</td>
<td>3,200,000</td>
<td>4,700,000</td>
<td>6,500,000</td>
<td>7,400,000</td>
<td>8,600,000</td>
<td>8,000,000</td>
</tr>
</tbody>
</table>

No data was available in 1941 because of the war.

a. Can the London population data be accurately modeled by a linear, quadratic, or exponential function? Explain.

b. A logistic growth equation can be written in the form

\[ P(t) = \frac{a}{1 + e^{-b(t-c)}} \]

where \( a \), \( b \), and \( c \) are positive numbers and \( t \) represents time measured in years. Using the application supplied, determine if the London population data can be accurately modeled by a logistic equation.

c. Explain the shape of the graph of \( P \) in terms of the structure of the equation \( P(t) = \frac{a}{1 + e^{-b(t-c)}} \). What impact do the values of \( a \), \( b \), and \( c \) have on the graph of \( P \)?

Example: Solution (DOK 3)

One online source suggests that exploiting solar energy makes sense in an area that receives 9 kWh/m² of solar energy per day and does not make sense in an area that receives only 2 kWh/m² of solar energy per day. Does it make sense to exploit solar energy in Santa Rosa?

Example: Solution (DOK 3)
Here are 6 containers that are being filled with water at a constant rate, and 9 graphs that represent the height of the water in a container as a function of the volume of water in the container.

- Choose the graph that corresponds to each container.
- Explain your reasoning clearly.
- For the remaining 3 graphs, sketch what the containers could look like.

Example: Solution (video clips in link) (DOK 3)
In this activity we will investigate the relationship between different quantities that are shown in a series of 10 video clips. Each clip includes a real time part, half speed playback, and a possible solution. (This allows students to make an initial sketch, adjust their sketch, and validate their results.)

For each video clip, watch the first part showing the situation in real time and then stop. At that point, sketch a graph of the relationship demonstrated on the set of graphs provided. In each graph pay special attention to key features such as increasing and decreasing intervals, maximums and minimums, intercepts, and constant and variable rates of change.

After drawing the initial sketch, watch the second part of the video showing the situation in half speed to double check your graph and revise it as appropriate.

Compare and discuss your graph with a neighbor or a different group. When you have reached an agreement of what the graph should look like, watch the last part of the video to see a possible solution. Did it agree with your graph? What was different, what was similar?

q. Example: Solution (DOK 3)

Every morning at summer camp, one of the campers has to hoist the camp flag to the top of the flagpole.

For each graph below, describe the action of the flag raiser that might have led to this graph that shows the height of the flag as a function of time.
Example: **Solution** (DOK 3)
Every morning at summer camp, one of the campers has to hoist the camp flag to the top of the flagpole.

For each graph below, describe the action of the flag raiser that might have led to this graph that shows the height of the flag as a function of time. Is any situation more realistic than another? Why or why not?

s. Example: **Solution** (DOK 3)
California is in the fifth year of a drought. Towns have been asked to decrease water use across the state. The graph below shows the water storage in Lake Sonoma, a reservoir in Northern California from 1988, shortly after it was first filled, until February 1, 2015.

a. What questions do you have looking at the graph?

b. Describe some of the features of the graph.

c. What might explain these features? (Consider data collection, weather events and reservoir management decisions.)

d. Based on the given information, should the region that depends on Lake Sonoma for water use implement additional water conservation measures?

t. Example: Solution (DOK 3)

Here is a graph that somebody drew for the height of a ball over time. Write a story that could have led to the graph.

u. Example: Solution (DOK 3)
PART 1:

Below is an illustration of an aquarium. When the faucet is on, the water flows into the aquarium at a constant rate. When the plug is pulled out, the water drains at a constant rate (but slower than the faucet's rate). At various times some events happen that affect the water level and/or the rate at which the water level changes.

![Aquarium Illustration]

Draw a graph of the water level as a function of time for each of the following situations:

a. The aquarium is initially empty with the plug in, and water flows in at a constant rate for 10 minutes.

b. The aquarium is initially half full with the plug in. Nothing happens for 5 minutes, then somebody pulls the plug.

c. The aquarium is half full and then a bucket of water is dumped into the aquarium.

d. There is a rock in the half-filled aquarium for the first 7 minutes, then it is removed.

PART 2:

The graph below shows the height of the water in the aquarium over a 17-minute time interval. Write a story from the point of view of someone watching what is going on with the aquarium to produce the given graph. Be creative! Invent a story that fits the graph. (Or: Write a story and then have a classmate produce the graph that goes with it.)

![Graph of Water Level]

PART 3:

The graph given above is a simplified version of reality. What simplifying assumptions did we make when we created the graph?

v. Example: Solution (DOK 2)
Below are 4 verbal descriptions, 3 graphs, and 3 tables of values. Match each of the following descriptions with an appropriate graph and table of values. Create the missing graph and the missing table of values.

1. The weight of your jumbo box of cereal decreases by an equal amount every week.

2. The value of the bicycle depreciated rapidly at first, but its value declined more slowly as time went on.

3. The tennis ball is dropped off the roof of a skyscraper.

4. For a while it looked like the decline in profits was slowing down, but then they began declining more rapidly.

A.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>400</td>
<td>384</td>
<td>336</td>
<td>256</td>
<td>144</td>
<td>0</td>
</tr>
</tbody>
</table>

B.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>400</td>
<td>184</td>
<td>98</td>
<td>63</td>
<td>49</td>
<td>43</td>
</tr>
</tbody>
</table>

C.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>400</td>
<td>253</td>
<td>218</td>
<td>216</td>
<td>181</td>
<td>34</td>
</tr>
</tbody>
</table>

a.

b.
w. Example: Solution (DOK 3)

The graphs below show three different quantities recorded by the flight computer during a flight of a small airplane.

a. Make a guess what each graph might represent.

b. The quantities are altitude (in feet), vertical velocity (in feet/min) and propeller speed (in RPM, revolutions per minute). Which graph represents altitude vs. time? Which represents vertical velocity vs. time? Which represents propeller speed vs. time?

c. Match the graphs with these quantities and give several reasons why your answers make sense.

d. About 13 minutes into the flight there are some unusual features on the graphs. What might have happened during the flight at that point?

e. Tell a story for this flight.
x. Example: **Solution** (DOK 3)

This task is in the form of classroom instructions; it is not intended to be distributed to students.

Introduce the video below with “Watch this very brief video and observe as much of what is happening as possible.”

![Video](image)

(0:19)

**Suggested format for the questions below:** Table/desk groups of around 4; discuss each question with group, then select some responses for the whole class to hear and discuss.

**After showing:**

a. What are some things that you think you might be able to find on that plane’s instrument panel?

b. This flight was about 3½ minutes long. Graph your best guess for one of the quantities that you identified in #1 over the course of the entire flight (not just the portion in the brief video).

Here is the instrument panel during the same takeoff:

![Instrument panel](image)

(0:19)

As the above shot may be difficult to make out all the details, here is a clearer photo of the same model display, in a slightly different mode:
a. Attempt to identify as many of the instruments on the panel as you can.

The students might ask about some of the abbreviations. You should feel free to reveal if asked.

- IAS: indicated airspeed, read directly from the plane's airspeed instrument
- DIS: distance from center of the airport
- NM: nautical mile, approximately 1.151 mile
- Knots: nautical miles per hour, 1 knot is approximately 1.151 mph
- Track: orientation of the plane's track (0° means heading straight North)
- Heading: direction the plane's nose is pointing (may differ from track due to wind)
- OAT: outside air temperature (on the panel next to local time)

Be certain that students have figured out where to read the time off of the instrument panel. Show the instrument panel video again, posing the question below:

b. Watch the video again, this time to figure out what time it was when the plane lifted off. Convince me!
After groups decide on a time estimate, show the following video to check how close they got with their "liftoff" estimates.

c. Now work on making your graph from #2 more precise and accurate. First, add some units to your graph.
d. What are some pieces of information that would help you improve your graph?

Share any of the following that are requested:

- Units for altitude: feet
- Maximum altitude 990 feet
- Time of maximum altitude: 11:32:08-11:32:14
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function \( h(n) \) gives the number of person-hours it takes to assemble \( n \) engines in a factory, then the positive integers would be an appropriate domain for the function. \( \ast \) (F-IF.B.5) (DOK 1,2) (DOK 1,2)
a. Example: **Solution** (DOK 3)

Mike likes to canoe. He can paddle 150 feet per minute. He is planning a river trip that will take him to a destination about 30,000 feet upstream (that is, against the current). The speed of the current will work against the speed that he can paddle.

a. Let $s$ be the speed of the current in feet per minute. Write an expression for $r(s)$, the speed at which Mike is moving relative to the river bank, in terms of $s$.

b. Mike wants to know how long it will take him to travel the 30,000 feet upstream. Write an expression for $T(s)$, the time in minutes it will take, in terms of $s$.

c. What is the vertical intercept of $T$? What does this point represent in terms of Mike's canoe trip?

d. At what value of $s$ does the graph have a vertical asymptote? Explain why this makes sense in the situation.

e. For what values of $s$ does $T(s)$ make sense in the context of the problem?

b. Example: **Solution** (DOK 3)

John makes DVDs of his friend's shows. He has realized that, because of his fixed costs, his average cost per DVD depends on the number of DVDs he produces. The cost of producing $x$ DVDs is given by

$$C(x) = 2500 + 1.25x.$$  

a. John wants to figure out how much to charge his friend for the DVDs. He's not trying to make any money on the venture, but he wants to cover his costs. Suppose John made 100 DVDs. What is the cost of producing this many DVDs? How much is this per DVD?

b. John is hoping to make many more than 100 DVDs for his friends. Complete the table showing his costs at different levels of production.

<table>
<thead>
<tr>
<th># of DVDs</th>
<th>0</th>
<th>10</th>
<th>100</th>
<th>1,000</th>
<th>10,000</th>
<th>100,000</th>
<th>1,000,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Cost</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost per DVD</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Explain why the average cost per DVD levels off.

d. Find an equation for the average cost per DVD of producing $x$ DVDs.

e. Find the domain of the average cost function.

f. Using the data points from your table above, sketch the graph of the average cost function. How does the graph reflect that the average cost levels off?

c. Example: **Solution** (DOK 3)
Mike likes to canoe. He can paddle 150 feet per minute. He is planning a river trip that will take him to a destination about 30,000 feet upstream (that is, against the current). The speed of the current will work against the speed that he can paddle.

a. Mike guesses that the current is flowing at a speed of 50 feet per minute. Assuming this is correct, how long will it take for Mike to reach his destination?

b. Mike does not really know the speed of the current. Make a table showing the time it will take him to reach his destination for different speeds:

<table>
<thead>
<tr>
<th>Speed of Current (feet per minute)</th>
<th>Mike's Speed (feet per minute)</th>
<th>Time for Mike to travel 30,000 feet (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>149</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>150</td>
<td></td>
</tr>
</tbody>
</table>

c. The time $T$ taken by the trip, in minutes, as a function of the speed of the current is $a$ feet/minute. Write an equation expressing $T$ in terms of $a$. Explain why $a = 150$ does not make sense for this function, both in terms of the canoe trip and in terms of the equation.

d. Sketch a graph of the equation in part (c). Explain why it makes sense that the graph has a vertical asymptote at $a = 150$.

d. Example: Solution (DOK 2)

Oakland Coliseum, home of the Oakland Raiders, is capable of seating 63,026 fans. For each game, the amount of money that the Raiders’ organization brings in as revenue is a function of the number of people, $n$, in attendance. If each ticket costs $30.00, find the domain and range of this function.

e. Example: Solution (DOK 2)

A restaurant is open from 2 pm to 2 am on a certain day, and a maximum of 200 clients can fit inside. If $f(t)$ is the number of clients in the restaurant $t$ hours after 2 pm that day,

a. What is a reasonable domain for $f$?

b. What is a reasonable range for $f$?

f. Example: SPEEDING TICKETS
New York State wants to change its system for assigning speeding fines to drivers. The current system allows a judge to assign a fine that is within the ranges shown in Table 1.

<table>
<thead>
<tr>
<th>Miles per Hour over Speed Limit</th>
<th>Minimum Fine</th>
<th>Maximum Fine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 10</td>
<td>$45</td>
<td>$150</td>
</tr>
<tr>
<td>11 - 30</td>
<td>$90</td>
<td>$300</td>
</tr>
<tr>
<td>31 or more</td>
<td>$180</td>
<td>$600</td>
</tr>
</tbody>
</table>

Some people have complained that the New York speeding fine system is not fair. The New Drivers Association (NDA) is recommending a new speeding fine system. The NDA is studying the Massachusetts system because of claims that it is fairer than the New York system.

![Table 2. Massachusetts Speeding Fines](image)

In this task, you will:

- Analyze the speeding fine systems for both New York and Massachusetts.
- Use data to propose a fairer speeding fine system for New York State.

The NDA claims that the proposed new model for the New York speeding fine system is fairer than the current system.

Do you agree or disagree with the claim? Explain your reasoning using specific examples from this task.

<table>
<thead>
<tr>
<th>Item</th>
<th>Claim</th>
<th>Domain</th>
<th>Target</th>
<th>DOK</th>
<th>CONTENT</th>
<th>MP</th>
<th>Key</th>
</tr>
</thead>
<tbody>
<tr>
<td>#7</td>
<td>4</td>
<td>NS</td>
<td>4A, 4B, 4D, 4F</td>
<td>3</td>
<td>HSF.IF.B.5</td>
<td></td>
<td>See exemplar</td>
</tr>
</tbody>
</table>

For this item, a full-credit response (2 points) includes
• agreeing with the claim
  AND

• Justifying the response by citing at least one comparison between values used in the two systems.
  For example,
• “I agree. In the current system, a driver who is ticketed for speeding by 11 mph could be fined $300. A driver who is ticketed for speeding by 30 mph could be fined $90. In the new system, any driver who speeds by 11 mph would pay $112 and a driver who speeds by 30 mph would pay $280. It is fairer that drivers who speed by the same amount will pay the same fine and the fine will increase as the excess speed increases.”
  For this item, a partial-credit response (1 point) includes:

• agreeing with the claim
  AND

• justifying the response WITHOUT citing any examples
  OR

• Justifying the incorrect response by citing examples from previous incorrect work in any of the previous items.
  For example,
• “I agree. It is fairer that drivers who are ticketed for the same excess speed will pay the same fine and the fine will increase as the excess speed increases.”
  For this item, a no-credit response (0 points) includes none of the features of a partial- or full-credit response.

  For example,

• “I agree.”
  This item is not graded on spelling or grammar.

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*
(F-IF.B.6)
  a. Example: Solution (DOK 3)
The school assembly is being held over the lunch hour in the school gym. All the teachers and students are there by noon and the assembly begins. About 45 minutes after the assembly begins, the temperature within the gym remains a steady 77 degrees Fahrenheit for a few minutes. As the students leave after the assembly ends at the end of the hour, the gym begins to slowly cool down.

Let $T$ denote the temperature of the gym in degrees Fahrenheit and $M$ denote the time, in minutes, since noon.

a. Is $M$ a function of $T$? Explain why or why not.

b. Explain why $T$ is a function of $M$, and consider the function $T = g(M)$. Interpret the meaning of $g(0)$ in the context of the problem.

c. Becky says: “The temperature increased 5 degrees in the first half hour after the assembly began.” Which of the following equations best represents this statement? Explain your choice.

i. $g(30) = 5$

ii. $\frac{g(30) - g(0)}{30} = 5$

iii. $g(30) - g(0) = 5$

iv. $T = g(30) - 5$

d. Which of these choices below represents the most reasonable value for the quantity $\frac{g(75) - g(60)}{15}$? Explain your choice:

i. 4

ii. 0.3

iii. 0

iv. -0.2

v. -5

b. Example: Solution (DOK 3)
You are a marine biologist working for the Environmental Protection Agency (EPA). You are concerned that the rare coral mathemafish population is being threatened by an invasive species known as the fluted dropout shark. The fluted dropout shark is known for decimating whole schools of fish. Using a catch-tag-release method, you collected the following population data over the last year.

<table>
<thead>
<tr>
<th># months since 1st measurement</th>
<th>Mathemafish population</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>480</td>
</tr>
<tr>
<td>1</td>
<td>472</td>
</tr>
<tr>
<td>2</td>
<td>417</td>
</tr>
<tr>
<td>3</td>
<td>318</td>
</tr>
<tr>
<td>4</td>
<td>240</td>
</tr>
<tr>
<td>5</td>
<td>152</td>
</tr>
<tr>
<td>6</td>
<td>103</td>
</tr>
<tr>
<td>7</td>
<td>94</td>
</tr>
<tr>
<td>8</td>
<td>47</td>
</tr>
<tr>
<td>9</td>
<td>32</td>
</tr>
<tr>
<td>10</td>
<td>24</td>
</tr>
<tr>
<td>11</td>
<td>29</td>
</tr>
<tr>
<td>12</td>
<td>46</td>
</tr>
</tbody>
</table>

Through intervention, the EPA was able to reduce the dropout population and slow the decimation of the mathemafish population. Your boss asks you to summarize the effects of the EPA's intervention plan in order to validate funding for your project.

What to include in your summary report:

- Calculate the average rate of change of the mathemafish population over specific intervals. Indicate how and why you chose the intervals you chose.
- When was the population decreasing the fastest?
- During what month did you notice the largest effects of the EPA intervention?
- Explain the overall effects of the intervention.
- Remember to justify all your conclusions using supporting evidence.

c. Example: **Solution** (DOK 2)
The table below shows the temperature, \( T \), in Tucson, Arizona \( t \) hours after midnight.

When does the temperature decrease the fastest: between midnight and 3 a.m. or between 3 a.m. and 4 a.m.?

<table>
<thead>
<tr>
<th>( t ) (hours after midnight)</th>
<th>0</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T ) (temp. in °F)</td>
<td>85</td>
<td>76</td>
<td>70</td>
</tr>
</tbody>
</table>

d. Example: **Solution** (DOK 3)

Jerry forgot to plug in his laptop before he went to bed. He wants to take the laptop to his friend's house with a full battery. The pictures below show screenshots of the battery charge indicator after he plugs in the computer.

![Battery charge screenshots](image)

a. When can Jerry expect that his laptop battery is fully charged?

b. At 9:27 AM Jerry makes a quick calculation:

\[
\text{The battery seems to be charging at a rate of 1 percentage point per minute. So the battery should be fully charged at 10:11 AM.}
\]

Explain Jerry's calculation. Is his estimate most likely an under- or over-estimate? How does it compare to your prediction?

c. Compare the average rate of change of the battery charging function on the first given time interval and on the last given time interval. What does this tell you about how the battery is charging?

d. How long would it take for the battery to charge if it started out completely empty?

e. Example: **Solution** (DOK 3)
A new internet organization called Illuminated Manifolds (IM) is trying to generate 2,000 insightful math problems spanning the K-12 curriculum in a timeframe of 48 months.

For integers $t$ with $0 \leq t \leq 32$, let $N(t)$ denote the total number of problems IM has generated by time $t$, measured in months since the beginning of September 2011. Suppose we are given the following information about $N(t)$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$N(t)$</th>
<th>$t$</th>
<th>$N(t)$</th>
<th>$t$</th>
<th>$N(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>12</td>
<td>514</td>
<td>24</td>
<td>879</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>14</td>
<td>590</td>
<td>26</td>
<td>901</td>
</tr>
<tr>
<td>4</td>
<td>42</td>
<td>16</td>
<td>666</td>
<td>28</td>
<td>922</td>
</tr>
<tr>
<td>6</td>
<td>163</td>
<td>18</td>
<td>744</td>
<td>30</td>
<td>943</td>
</tr>
<tr>
<td>8</td>
<td>400</td>
<td>20</td>
<td>822</td>
<td>32</td>
<td>1000</td>
</tr>
<tr>
<td>10</td>
<td>438</td>
<td>22</td>
<td>868</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Let $R(t)$ denote the remaining number of problems the organization still has to construct at time $t$. Find an algebraic relation between $N(t)$ and $R(t)$ -- how does this relation reveal itself upon graphing the two functions?

b. Let $B(t)$ denote the number of problems produced in the $t$-th month. Using the information from the table, explain how you can approximate $B(t)$ for $0 \leq t \leq 32$. Were there any extraordinarily productive spurts in IM’s timeline?

c. At any time $t$, let $S(t)$ denote the average number of problems that IM has to write per month to meet its goal by the deadline. Find an algebraic expression for $S$ in terms of $N$. Find some interesting values of $S(t)$.

d. Using all of the data collected so far, decide whether or not you think IM will achieve its goal, and provide a justification for your prediction. Discuss what additional data you would like to have to better inform your prediction.

f. Example: The height of a plant, in centimeters, is modeled as a function of time, in days. Consider this graph of the function.
Write the average rate of change for the height of the plant, measured in centimeters per day, between day 0 and day 20.

<table>
<thead>
<tr>
<th>Item</th>
<th>Claim</th>
<th>Domain</th>
<th>Target</th>
<th>DOK</th>
<th>CONTENT</th>
<th>MP</th>
<th>Key</th>
</tr>
</thead>
<tbody>
<tr>
<td>#15</td>
<td>1</td>
<td>F-IF</td>
<td>L</td>
<td>2</td>
<td>HSF.IF.B.6</td>
<td>2</td>
<td>1 to 1.4</td>
</tr>
</tbody>
</table>
Analyze functions using different representations (F-IF.C)

**Example:** The graphs of $y = g(x)$ and $y = f(x)$ are shown. Add a point that will satisfy each given condition.

A point on the graph of $g$ where $x = 0$
A point on the graph of $g$ where $f(x) > g(x)$
A point on the graph of $f$ where $f(x) = 0$

**Example:** The graph of $y = x^2$ is shown on the grid.

Draw on the grid to show the graph of $y = (x - 4)^2 + 2$.

<table>
<thead>
<tr>
<th>Item</th>
<th>Claim</th>
<th>Domain</th>
<th>Target</th>
<th>DOK</th>
<th>CONTENT</th>
<th>MP</th>
<th>Key</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>1</td>
<td>F-IF, F-BF</td>
<td>M</td>
<td>2</td>
<td>HSF.IF.C, HSF.BF.B.3</td>
<td>N/A</td>
<td></td>
</tr>
</tbody>
</table>

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

- Graph linear and quadratic functions and show intercepts, maxima, and minima.
- Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
- Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.
- (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.
- Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ([F-IF.C.7](DOK 1,2))

1. Example: [Solution](DOK 3)
a. Graph these equations on your graphing calculator at the same time. What happens? Why?

\[ y_1 = (x-3)(x + 1) \]
\[ y_2 = x^2-2x-3 \]
\[ y_3 = (x-1)^2-4 \]
\[ y_4 = x^2-2x+1 \]

b. Below are the first three equations from the previous problem.

\[ y_1 = (x-3)(x + 1) \]
\[ y_2 = x^2-2x-3 \]
\[ y_3 = (x-1)^2-4 \]

These three equations all describe the same function. What are the coordinates of the following points on the graph of the function? From which equation is each point most easily determined? Explain.

i. vertex: _____
ii. y-intercept: _____
iii. x-intercept(s): _____

c. Make up an equation for a quadratic function whose graph satisfies the given condition. Use whatever form is most convenient.

i. Has a vertex at \((-2, -5)\).
ii. Has a y-intercept at \((0, 6)\)
iii. Has x-intercepts at \((3, 0)\) and \((5, 0)\)
iv. Has x-intercepts at the origin and \((4, 0)\)
v. Goes through the points \((4, 2)\) and \((1, 2)\)

2. Example: **Solution** (DOK 2)

a. Sketch the graphs of the functions described by \(f(x) = x^2\) and \(g(x) = x^4\) on the same axes, being careful to label any points of intersection. Also, find and label \(\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)\) and \(\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)\).

b. Sketch the graphs of the functions described by \(f(x) = x^3\) and \(g(x) = x^5\) on the same axes, being careful to label any points of intersection. Also, find and label \(\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)\) and \(\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)\).

c. Sketch the graphs of the functions described by \(f(x) = x^2\) and \(g(x) = x^3\) on the same axes, being careful to label any points of intersection. Also, find and label \(\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)\) and \(\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)\).
3. **Example:** Solution (DOK 2)
Consider the following four functions

- \( f(x) = \frac{3}{1 + e^{-3x}} \)
- \( g(x) = 1 - \frac{e^{-x}}{2} \)
- \( h(x) = -2 + \frac{e^x}{2} \)
- \( k(x) = \frac{3}{1 + e^{3x}} \)

Below are four graphs of functions shown for \(-2 \leq x \leq 2\). Match each function with its graph and explain your choice:

4. **Example:** Solution (DOK 2)

Many computer applications use very complex mathematical algorithms. The faster the algorithm, the more smoothly the programs run. The running time of an algorithm depends on the total number of steps needed to complete the algorithm. For image processing, the running time of an algorithm increases as the size of the image increases.

For an \( n \times n \) image, algorithm 1 has running time given by \( p(n) = n^3 + 3n + 1 \) and algorithm 2 has running time given by \( q(n) = 15n^2 + 5n + 4 \) (measured in nanoseconds, or \( 10^{-9} \) seconds).

a. Compute the running time for both algorithms for images of size 10-by-10 pixels and 100-by-100 pixels.

b. Graph both running time polynomials in an appropriate window (or several windows if necessary).

c. Which algorithm is more efficient?
5. Example: **Solution** (DOK 3)

The table below shows historical estimates for the population of London.

<table>
<thead>
<tr>
<th>Year</th>
<th>1801</th>
<th>1821</th>
<th>1841</th>
<th>1861</th>
<th>1881</th>
<th>1901</th>
<th>1921</th>
<th>1939</th>
<th>1961</th>
</tr>
</thead>
<tbody>
<tr>
<td>London population</td>
<td>1,100,000</td>
<td>1,600,000</td>
<td>2,200,000</td>
<td>3,200,000</td>
<td>4,700,000</td>
<td>6,500,000</td>
<td>7,400,000</td>
<td>8,600,000</td>
<td>8,000,000</td>
</tr>
</tbody>
</table>

No data was available in 1941 because of the war.

a. Can the London population data be accurately modeled by a linear, quadratic, or exponential function? Explain.

b. A **logistic growth** equation can be written in the form

$$P(t) = \frac{a}{1 + e^{-k(t-c)}}$$

where $a$, $b$, and $c$ are positive numbers and $t$ represents time measured in years. Using the application supplied, determine if the London population data can be accurately modeled by a logistic equation.

c. Explain the shape of the graph of $P$ in terms of the structure of the equation $P(t) = \frac{a}{1 + e^{-kt}}$. What impact do the values of $a$, $b$, and $e$ have on the graph of $P$?

6. Example: **Solution** (DOK 2)

In this task we are going to investigate the graphs of $f(x) = \frac{1}{x + a}$ and $g(x) = \frac{1}{x^2 + b}$. Move the sliders below to change the values of $a$ and $b$.

![Graph of functions](image)

a. Describe your observations.

b. Connect the features you observed on the graphs to the structure of the expressions of the functions.

7. Example: **Solution** (DOK 2)
Below is a picture of the functions \( f(x) = \log_b x \) and \( g(x) = b^x \). In the application below, the base \( b \) varies between 1 and 2 (by hundredths) and pressing the "play" button will run through all possible values of \( b \).

8. Example: Solution (DOK 3)

At the beginning of the week, Jessie had $500 in her bank account. She deposited a check for $50 on Tuesday and then paid $250 in rent on Wednesday. On Friday, Jessie deposited $200 in the account and then on Saturday she paid $50 for groceries from her bank account. Jessie made the following graph for the balance in her bank account during this week:

a. Is the depiction of how the account balance varies over the week accurate? Explain.

b. How can Jessie graphically represent the bank account balance in a way that better shows how it changes?
9. Example: Solution (DOK 3)

An exponential function is a function of the form \( f(x) = ab^x \) for positive real numbers \( a \) and \( b \).

a. Use the app below to sketch exponential functions for various values of \( a \) and \( b \). Describe in words the effect of changing \( a \) and \( b \) on the shape of the graph.

b. Find a function of the form \( f(x) = a \cdot b^x \) for each of the four graphs below.

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   b. Use the properties of exponents to interpret expressions for exponential functions. For example,
identify percent rate of change in functions such as \( y = (1.02)^t \), \( y = (0.97)^t \), \( y = (1.01)^{12t} \), \( y = (1.2)^{\frac{t}{10}} \), and classify them as representing exponential growth or decay. (F-IF.C.8) (DOK 1,2)

1. Example: Solution (DOK 3)

Suppose \( h(t) = -5t^2 + 10t + 3 \) is the height of a diver above the water (in meters), \( t \) seconds after the diver leaves the springboard.

a. How high above the water is the springboard? Explain how you know.

b. When does the diver hit the water?

c. At what time on the diver's descent toward the water is the diver again at the same height as the springboard?

d. When does the diver reach the peak of the dive?

2. Example: Solution (DOK 3)

Which of the following could be the function of a real variable \( x \) whose graph is shown below? Explain.

\[
\begin{align*}
  f_1(x) &= (x + 12)^2 + 4 \\
  f_2(x) &= -(x - 2)^2 - 1 \\
  f_3(x) &= -(x + 18)^2 - 40 \\
  f_4(x) &= (x - 12)^2 - 9 \\
  f_5(x) &= -(x + 4)(x + 3) \\
  f_6(x) &= -(x + 4)(x - 6) \\
  f_7(x) &= (x - 12)(x + 18) \\
  f_8(x) &= (24 - x)(40 - x)
\end{align*}
\]

3. Example: Solution (DOK 2)

A preserved plant is estimated to contain 1 microgram (a millionth of a gram) of Carbon 14. The amount of Carbon 14 present in the preserved plant is modeled by the equation

\[ f(t) = A \left( \frac{1}{2} \right)^{\frac{t}{5730}} \]

where \( t \) denotes time since the death of the plant, measured in years, and \( A \) is the amount of Carbon 14 present in the plant at death, measured in micrograms.

a. How much Carbon 14 was present in the living plant assuming it died 5000 years ago?

b. How much Carbon 14 was present in the living plant assuming it died 10000 years ago?

c. The half-life of Carbon 14 is the amount of time it takes for half of the Carbon 14 to decay. What half-life does the expression for the function \( f \) imply for Carbon 14?
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum. (F-IF.C.9) (DOK 1,2)
   a. Example: Solution (DOK 3)

   Suppose Brett and Andre each throw a baseball into the air. The height of Brett's baseball is given by
   
   \[ h(t) = -16t^2 + 79t + 6, \]
   
   where \( h \) is in feet and \( t \) is in seconds. The height of Andre's baseball is given by the graph below:

   [Graph of a quadratic function]

   Brett claims that his baseball went higher than Andre's, and Andre says that his baseball went higher.

   a. Who is right?
   b. How long is each baseball airborne?
   c. Construct a graph of the height of Brett's throw as a function of time on the same set of axes as the graph of Andre's throw (if not done already), and explain how this can confirm your claims to parts (a) and (b).

---

**Building Functions**

Build a function that models a relationship between two quantities (F-FB.A)

1. Write a function that describes a relationship between two quantities. *
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
   b. Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.
   c. (+) Compose functions. For example, if \( T(y) \) is the temperature in the atmosphere as a function of height, and \( h(t) \) is the height of a weather balloon as a function of time, then \( T(h(t)) \) is the temperature at the location of the weather balloon as a function of time. (F-FB.A.1) (DOK 1,2)

   1. Example: Solution (DOK 3)
You have been hired for a summer internship at a marine life aquarium. Part of your job is diluting brine for the saltwater fish tanks. The brine is composed of water and sea salt, and the salt concentration is 15.8\% by mass, meaning that in any amount of brine the mass of salt is 15.8\% of the total mass.

a. The supervisor asks you to add fresh water to one liter of the brine using a half-liter measuring cup. Let \( S(x) \) be the salt concentration of the resulting mixture when you add \( x \) half-liters of salt. Write an expression for \( S(x) \). [Assume that one liter of water has mass 1 kg.]

b. Describe how the graph of \( S \) is related to the graph of \( y = \frac{1}{x} \).

c. Sketch the graph of \( S \).

d. How much fresh water should you add to get a mixture which is 4\% sea salt, approximately the salt concentration of the ocean?

2. Example: Solution (DOK 2)

(a) How many cubes are needed to build this tower?

(b) How many cubes are needed to build a tower like this, but 12 cubes high? Justify your reasoning.

c. How would you calculate the number of cubes needed for a tower \( n \) cubes high?

3. Example: Solution (DOK 2)
At the beginning of January, Susita had some money in her checking account. At the end of each month she deposits enough to double the amount currently in the account. However, she has a loan to pay off, requiring her to withdraw $10 from the account monthly (immediately after her deposit).

a. Assuming January is the first month, write an equation that describes the amount of money in Susita’s account at the end of the $n^{th}$ month, $S(n)$, in terms of the amount of money in Susita’s account at the end of the $(n - 1)^{th}$ month, $S(n - 1)$.

b. At the end of May, Susita had $2 left in the account. How much did she have at the end of January?

4. Example: Solution (DOK 2)

Using the graphs below, sketch a graph of the function $s(x) = f(x) + g(x)$.

5. Example: Solution (DOK 2)
According to the U.S. Energy Information Administration, a barrel of crude oil produces approximately 20 gallons of gasoline. EPA mileage estimates indicate a 2011 Ford Focus averages 28 miles per gallon of gasoline.

a. Write an expression for $G(x)$, the number of gallons of gasoline produced by $x$ barrels of crude oil.

b. Write an expression for $M(g)$, the number of miles on average that a 2011 Ford Focus can drive on $g$ gallons of gasoline.

c. Write an expression for $M(G(x))$. What does $M(G(x))$ represent in terms of the context?

d. One estimate (from www.oilvoice.com) claimed that the 2010 Deepwater Horizon disaster in the Gulf of Mexico spilled 4.9 million barrels of crude oil. How many miles of Ford Focus driving would this spilled oil fuel?

6. Example: Solution (DOK 3)

Kimi and Jordan are each working during the summer to earn money in addition to their weekly allowance. Kimi earns $9 per hour at her job, and her allowance is $8 per week. Jordan earns $7.50 per hour, and his allowance is $16 per week.

a. Jordan wonders who will have more income in a week if they both work the same number of hours. Kimi says, “It depends.” Explain what she means.

b. Is there a number of hours worked for which they will have the same income? If so, find that number of hours. If not, why not?

c. What would happen to your answer to part (b) if Kimi were to get a raise in her hourly rate? Explain.

d. What would happen to your answer to part (b) if Jordan were no longer to get an allowance? Explain.

7. Example: Solution (DOK 3)

Let $f$ be the function that assigns to a temperature in degrees Celsius its equivalent in degrees Fahrenheit. Let $g$ be the function that assigns to a temperature in degrees Kelvin its equivalent in degrees Celsius.

a. Explain what $x$ and $f(g(x))$ represent in terms of temperatures, or explain why there is no reasonable representation.

b. Explain what $x$ and $g(f(x))$ represent in terms of temperatures, or explain why there is no reasonable representation.

c. Given that $f(x) = \frac{9}{5}x + 32$ and $g(x) = x - 273$, find an expression for $f(g(x))$.

d. Find an expression for the function $h$ which assigns to a temperature in degrees Fahrenheit its equivalent in degrees Kelvin.

8. Example: Solution (DOK 3)
Mike likes to canoe. He can paddle 150 feet per minute. He is planning a river trip that will take him to a destination about 30,000 feet upstream (that is, against the current). The speed of the current will work against the speed that he can paddle.

a. Let $s$ be the speed of the current in feet per minute. Write an expression for $r(s)$, the speed at which Mike is moving relative to the river bank, in terms of $s$.

b. Mike wants to know how long it will take him to travel the 30,000 feet upstream. Write an expression for $T(s)$, the time in minutes it will take, in terms of $s$.

c. What is the vertical intercept of $T$? What does this point represent in terms of Mike's canoe trip?

d. At what value of $s$ does the graph have a vertical asymptote? Explain why this makes sense in the situation.

e. For what values of $s$ does $T(s)$ make sense in the context of the problem?

9. Example: **Solution** (DOK 3)

Mike likes to canoe. He can paddle 150 feet per minute. He is planning a river trip that will take him to a destination about 30,000 feet upstream (that is, against the current). The speed of the current will work against the speed that he can paddle.

a. Mike guesses that the current is flowing at a speed of 50 feet per minute. Assuming this is correct, how long will it take for Mike to reach his destination?

b. Mike does not really know the speed of the current. Make a table showing the time it will take him to reach his destination for different speeds:

<table>
<thead>
<tr>
<th>Speed of Current (feet per minute)</th>
<th>Mike's Speed (feet per minute)</th>
<th>Time for Mike to travel 30,000 feet (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>140</td>
<td></td>
<td></td>
</tr>
<tr>
<td>149</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. The time $T$ taken by the trip, in minutes, as a function of the speed of the current is $s$ feet/minute. Write an equation expressing $T$ in terms of $s$. Explain why $s = 150$ does not make sense for this function, both in terms of the canoe trip and in terms of the equation.

d. Sketch a graph of the equation in part (c). Explain why it makes sense that the graph has a vertical asymptote at $s = 150$.  

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10. Example: Solution (DOK 3)

On June 1, a fast growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.

a. When will the lake be covered halfway?

b. On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?

c. On June 29, a clean-up crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?

d. Write an equation that represents the percentage of the surface area of the lake that is covered in algae as a function of time (in days) that passes since the algae was introduced into the lake if the cleanup crew does not come on June 29.

11. Example: Solution (DOK 2)

A man invests $1000 in an account with a 5% annual interest rate. He knows that money in an account where interest is compounded semi-annually will earn interest faster than money in an account where interest is compounded annually. He wonders how much interest can be earned by compounding it more and more often. In this problem we investigate his question.

If the man's interest is compounded annually, his year-end balance will be:

\[
\begin{align*}
1000 + 5\% \cdot 1000 &= 1000 + 0.05 \cdot 1000 \\
&= 1000(1 + 0.05) \\
&= 1050.
\end{align*}
\]

If his interest is compounded semi-annually, he earns half the annual interest at mid-year, and his mid-year balance is:

\[
\begin{align*}
1000 + \frac{5\%}{2} \cdot 1000 &= 1000 + \frac{0.05}{2} \cdot 1000 \\
&= 1000 \left( 1 + \frac{0.05}{2} \right) \\
&= 1025.
\end{align*}
\]

At year-end he earns the other half of his annual interest and his year-end balance is:
\[
\begin{align*}
1025 + \frac{5\%}{2} \cdot 1025 &= \frac{0.05}{2} \cdot 1025 \\
&= 1025 \left(1 + \frac{0.05}{2}\right) \\
&= 1000 \left(1 + \frac{0.05}{2}\right)^2 \\
&= 1050.625.
\end{align*}
\]

a. Find the end of year balance if the interested is compounded quarterly.

b. Write an expression which gives the man’s end of year balance in terms of the number of times the interest is compounded, n.

c. Substitute \( k = \frac{0.05}{n} \) into your expression so that the whole expression is written in terms of \( k \) instead of in terms of \( n \).

d. Now we’ll investigate what happens to the end of year balance as we compound the interest more and more. This means that we want to increase the value of \( n \). What does increasing the value of \( n \) do to the value of \( k \)?

e. Complete the table below to help you see what happens to the end of year balance as \( k \) becomes larger and larger. Round to the 5th decimal place.

<table>
<thead>
<tr>
<th>( k )</th>
<th>((1 + k)^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>( (1 + 0.1)^2 )</td>
</tr>
<tr>
<td>0.01</td>
<td>( (1 + 0.01)^2 )</td>
</tr>
<tr>
<td>0.001</td>
<td>( (1 + 0.001)^2 )</td>
</tr>
<tr>
<td>0.0001</td>
<td>( (1 + 0.0001)^2 )</td>
</tr>
<tr>
<td>0.00001</td>
<td>( (1 + 0.00001)^2 )</td>
</tr>
</tbody>
</table>

The values in the second column of your table should not appear to be growing out of control. They should appear to approach a limiting value. This value is an irrational number which mathematicians denote with the letter \( e \).

f. Based on the results of your table, what value does it appear the end of year balance will approach as the interest is compounded more and more often? Write this value in terms of \( e \).

12. Example: Solution (DOK 3)
A man knows that money in an account where interest is compounded semi-annually will earn interest faster than money in an account where interest is compounded annually. He wonders how much interest can be earned by compounding it more and more often. In this problem we investigate his question.

For ease of computation, let's suppose the man invests $1 at a 100% interest rate. If his interest is compounded annually, his year-end balance will be:

\[ \$1 + 100\% \cdot \$1 = \$1 + 1.00 \cdot \$1 = \$1(1 + 1) = \$2. \]

If his interest is compounded semi-annually, he earns half the annual interest at mid-year, and so his mid-year balance is:

\[ \$1 + \frac{100\%}{2} \cdot \$1 = \$1 + \frac{1.00}{2} \cdot \$1 = \$1 \cdot \left(1 + \frac{1}{2}\right) = \$1.5. \]

At year-end he earns the other half of his annual interest giving him a year-end balance of:

\[ \$1.5 + \frac{100\%}{2} \cdot \$1.5 = \$1.5 + \frac{1.00}{2} \cdot \$1.5 = \$1.5 \cdot \left(1 + \frac{1}{2}\right) = \$1 \cdot \left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{2}\right) = \$1 \cdot \left(1 + \frac{1}{2}\right)^2 = \$2.25. \]

a. Find the man's year-end balance if his interest is compounded quarterly.

b. Write an expression which gives the man's year-end balance in terms of the number of times the interest is compounded, \( n \).

c. Now we'll investigate what happens to the year-end balance as we compound the interest more and more. This means that we want to increase the value of \( n \). Complete the table below to help you see what happens to the end of year balance as \( n \) becomes larger and larger. Round values to the 5th decimal place.
d. Based on the results of your table, does it appear that the man can make an unlimited amount of money off of his $1 investment if the bank compounds the interest more and more often? Explain.

13. Example: Solution (DOK 3)

For each function in this task, assume the domain is the largest set of real numbers for which the function value is a real number.

Let \( f \) be the function defined by \( f(x) = x^2 \). Let \( g \) be the function defined by \( g(x) = \sqrt{x} \).

a. Sketch the graph of \( y = f(g(x)) \) and explain your reasoning.

b. Sketch the graph of \( y = g(f(x)) \) and explain your reasoning.

14. Example: Solution (DOK 3)

Suppose the swine flu, influenza H1N1, is spreading on a school campus. The following table shows the number of students, \( n \), that have the flu \( d \) days after the initial outbreak. The number of students who have the flu is a function of the number of days, \( n = f(d) \).

<table>
<thead>
<tr>
<th>( d ) (days) ( n = f(d) ) (number of students infected)</th>
<th>0</th>
<th>2</th>
<th>6</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>9</td>
<td>16</td>
<td>30</td>
<td>55</td>
<td>45</td>
<td>32</td>
<td></td>
</tr>
</tbody>
</table>

There is a school store on campus. As the number of students who have the flu increases, the number of tissue boxes, \( b \), sold at the school store also increases. The number of tissue boxes sold on a given day is a function of the number of students who have the flu, \( b = g(n) \), on that day.

<table>
<thead>
<tr>
<th>( n ) (number of students infected) ( b = g(n) ) (number of tissue boxes sold)</th>
<th>0</th>
<th>3</th>
<th>8</th>
<th>9</th>
<th>12</th>
<th>16</th>
<th>18</th>
<th>30</th>
<th>32</th>
<th>38</th>
<th>45</th>
<th>50</th>
<th>55</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>12</td>
<td>13</td>
<td>18</td>
<td>24</td>
<td>33</td>
<td>34</td>
<td>40</td>
<td>40</td>
<td>51</td>
<td>57</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Find \( g(f(0)) \) and state the meaning of this value in the context of the flu epidemic. Include units in your answer.

b. Fill in the chart below using the fact that \( b = g(f(d)) \).

<table>
<thead>
<tr>
<th>( d ) (days) ( b = g(f(d)) ) (number of tissue boxes sold)</th>
<th>0</th>
<th>2</th>
<th>6</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>24</th>
</tr>
</thead>
</table>
c. For each of the following expressions, explain its meaning in the context of the problem, and if possible, give an approximation of its value. Justify your answer.
   i. \( g(f(16)) \)
   ii. \( g(f(18)) \)
   iii. \( f(g(9)) \)

15. Example: Solution (DOK 3)

   Let \( f \) be the function defined by \( f(x) = 2x^2 + 4x - 16 \). Let \( g \) be the function defined by
   \[
   g(x) = 2(x + 1)^2 - 18.
   \]

   a. Verify that \( f(x) = g(x) \) for all \( x \).
   b. In what ways do the equivalent expressions \( 2x^2 + 4x - 16 \) and \( 2(x + 1)^2 - 18 \) help to understand the function \( f \)?
   c. Consider the functions \( h, l, m, \) and \( n \) given by
      \[
      \begin{align*}
      h(x) &= x^2 \\
      l(x) &= x + 1 \\
      m(x) &= x - 9 \\
      n(x) &= 2x
      \end{align*}
      \]
      Show that \( f(x) \) is a composition, in some order, of the functions \( h, l, m, \) and \( n \). How do you determine the order of composition?
   d. Explain the impact each of the functions \( l, m, \) and \( n \) has on the graph of the composition.

16. Example: Solution (DOK 3)
A new internet organization called Illuminated Manifolds (IM) is trying to generate 2,000 insightful math problems spanning the K–12 curriculum in a timeframe of 48 months.

For integers $t$ with $0 \leq t \leq 32$, let $N(t)$ denote the total number of problems IM has generated by time $t$, measured in months since the beginning of September 2011. Suppose we are given the following information about $N(t)$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$N(t)$</th>
<th>$t$</th>
<th>$N(t)$</th>
<th>$t$</th>
<th>$N(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>12</td>
<td>514</td>
<td>24</td>
<td>879</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>14</td>
<td>590</td>
<td>26</td>
<td>901</td>
</tr>
<tr>
<td>4</td>
<td>42</td>
<td>16</td>
<td>666</td>
<td>28</td>
<td>922</td>
</tr>
<tr>
<td>6</td>
<td>163</td>
<td>18</td>
<td>744</td>
<td>30</td>
<td>943</td>
</tr>
<tr>
<td>8</td>
<td>400</td>
<td>20</td>
<td>822</td>
<td>32</td>
<td>1000</td>
</tr>
<tr>
<td>10</td>
<td>438</td>
<td>22</td>
<td>868</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Let $B(t)$ denote the remaining number of problems the organization still has to construct at time $t$. Find an algebraic relation between $N(t)$ and $B(t)$ — how does this relation reveal itself upon graphing the two functions?

b. Let $B(t)$ denote the number of problems produced in the $t$-th month. Using the information from the table, explain how you can approximate $B(t)$ for $0 \leq t \leq 32$. Were there any extraordinarily productive spurts in IM's timeline?

c. At any time $t$, let $S(t)$ denote the average number of problems that IM has to write per month to meet its goal by the deadline. Find an algebraic expression for $S$ in terms of $N$. Find some interesting values of $S(t)$.

d. Using all of the data collected so far, decide whether or not you think IM will achieve its goal, and provide a justification for your prediction. Discuss what additional data you would like to have to better inform your prediction.

17. Example: The “two-second rule” is used by a driver who wants to maintain a safe following distance at any speed. A driver must count two seconds from when the car in front of him or her passes a fixed point, such as a tree, until the driver passes the same fixed point. Drivers use this rule to determine the minimum distance to follow a car traveling at the same speed. A diagram representing this distance is shown.
As the speed of the cars increases, the minimum following distance also increases. Explain how the “two-second rule” leads to a greater minimum following distance as the speed of the cars increases. As part of your explanation, include the minimum following distances, in feet, for cars traveling at 30 miles per hour and 60 miles per hour.

<table>
<thead>
<tr>
<th>Item</th>
<th>Claim</th>
<th>Domain</th>
<th>Target</th>
<th>DOK</th>
<th>CONTENT</th>
<th>MP</th>
<th>Key</th>
</tr>
</thead>
<tbody>
<tr>
<td>#6</td>
<td>4</td>
<td>NS</td>
<td>4E</td>
<td>2</td>
<td>F-BF.1a, F-LE.1b</td>
<td></td>
<td>See Below</td>
</tr>
</tbody>
</table>

Sample Top-Score Response:

The minimum following distance is determined by the formula $d=rt$ where $d$ is the minimum following distance, $r$ is the rate (or speed), and $t$ is the time. The “two-second rule” says that the time needed between cars traveling at the same speed remains constant at 2 seconds, so as the speed of the cars increases by a certain factor, then the minimum following distance must increase by the same factor. Since the speed of the cars is measured in miles per hour, and the “two-second rule” measures time in seconds, I used the formula shown below to determine the minimum following distance, in feet.

$$d = r \cdot \left( \frac{5280}{1} \right) \cdot \left( \frac{1}{3600} \right) \cdot 2$$

For cars traveling at 30 miles per hour, the minimum distance is 88 feet. For cars traveling at 60 miles per hour, the minimum following distance is 176 feet.

For full credit (2 points):

The response demonstrates a full and completed understanding of analyzing real-world scenarios. The response contains the following evidence:

- The student describes the correct relationship between speed and minimum following distance as proportional (e.g., as the speed increases, the minimum following distance needs to increase by the same factor).
AND

• The student determines the correct minimum following distances at 30 miles per hour and 60 miles per hour.

**For partial credit (1 point):**

The response demonstrates a partial understanding of communicating reasoning. The response contains the following evidence:

The student describes the correct relationship between speed and minimum following distance as proportional (e.g., as the speed increases, the minimum following distance needs to be increased by the same factor).

OR

• The student determines the correct minimum following distances at 30 miles per hour and 60 miles per hour.

OR

• The student determines an incorrect minimum following distance but provides correct relationship between speed and minimum following distance as proportional.

2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. * (F-B.A.2) (DOK 1,2)

a. Example: Solution (DOK 2)

In a video game called Snake, a player moves a snake through a square region in the plane, trying to eat the white pellets that appear.

If we imagine the playing field as a 32-by-32 grid of pixels, then the snake starts as a 4-by-1 rectangle of pixels, and grows in length as it eats the pellets:

• After the first pellet, it grows in length by one pixel.
• After the second pellet, it further grows in length by two pixels.
• After the third pellet, it further grows in length by three pixels.
• and so on, with the nth pellet increasing its length by n pixels.
Let $L(n)$ denote the length of the snake after eating $n$ pellets. For example, $L(3) = 10$.

a. How long is the snake after eating 4 pellets? After 5 pellets? After 6 pellets?

b. Find a recursive description of the function $L(n)$.

c. Find a non-recursive expression for $L(100)$, and evaluate that expression to compute $L(100)$.

d. What is the largest number of pellets a snake could eat before he could no longer fit in the playing field? That is, how long is a perfect game of snake?

Build new functions from existing functions (F-FB.B)

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of $k$ (both positive and negative); find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. (F-FB.B.3) (DOK 1,2)

a. Example: Solution (DOK 1)

Determine whether each of these functions is odd, even, or neither. Use algebraic methods on all of the functions. You may start out by looking at a graph, if you need to.

a. $f(x) = 3^x + 3^{-x}$

b. $g(x) = 2^x - 2^{-x}$

c. $h(x) = x^2 + 4x - 2$

d. $j(x) = x^3 - 4x$
b. Example: Solution (DOK 3)

In “Building an explicit quadratic function” a particular quadratic function was rewritten by completing the square. The quadratic function used was \( q(x) = 2x^2 + 4x - 16 \) and this function was rewritten as

\[
q(x) = 2(x + 1)^2 - 18 = 2((x + 1)^2 - 9).
\]

Some of the advantages to this form are that the x-coordinate of the vertex of the graph of \( q \) can be found more easily and the two roots of \( q \) can also be found readily. The right hand side of this equation can be seen as a horizontal translation by \(-1\), then squaring, then a vertical translation by \(-9\), and finally a multiplicative scaling by \(2\). The goal of this task is first to employ the same technique on a general quadratic function and then derive the quadratic formula. To assist in this process, we first rewrite the equation above:

\[
\frac{q(x)}{2} = (x + 1)^2 - 9.
\]

Let \( f \) be a quadratic function, so we have

\[
f(x) = ax^2 + bx + c.
\]

Here \( a, b, c \) are real numbers and we assume that \( a \) is non-zero.

a. Following the lead of our example problem, we begin by dividing out the leading coefficient:

Multiplying \( f(x) \) by \( \frac{1}{a} \) gives

\[
\frac{f(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}.
\]

We wish to write this as the square of a linear function \( x + d \) plus a constant. Find numbers \( d \) and \( k \) so that \( \frac{f(x)}{a} = (x + d)^2 + k \).

b. Using the values for \( d \) and \( k \) found in part (b), rewrite \( f(x) = ax^2 + bx + c \) as

\[
\frac{f(x)}{a} = (x + d)^2 + k.
\]

Why do the expressions \( f(x) \) and \( \frac{f(x)}{a} \) have the same roots?

c. Explain how to deduce the quadratic formula for the roots of \( f \) from part (c).
c. Example: Solution (DOK 2)

A computer game uses functions to simulate the paths of an archer's arrows. The z-axis represents the level ground on which the archer stands, and the coordinate pair \((2, 5)\) represents the top of a castle wall over which he is trying to fire an arrow.

In response to user input, the first arrow followed a path defined by the function \(f(x) = 6 - x^2\), failing to clear the castle wall.

![Graph of function](image)

The next arrow must be launched with the same force and trajectory, so the user must reposition the archer in order for his next arrow to have any chance of clearing the wall.

a. How much closer to the wall must the archer stand in order for the arrow to clear the wall by the greatest possible distance?

b. What function must the user enter in order to accomplish this?

c. If the user can only enter functions of the form \(f(x + k)\), what are all the values of \(k\) that would result in the arrow clearing the castle wall?

d. Example: Solution (DOK 3)
Suppose \( f(x) = x^2 \) where \( x \) can be any real number.

a. Sketch a graph of the function \( f \).

b. Sketch a graph of the function \( g \) given by
   \[ g(x) = f(x) + 2. \]
   How do the graphs of \( f \) and \( g \) compare? Why?

c. Sketch a graph of the function \( h \) given by
   \[ h(x) = -2 \cdot f(x). \]
   How do the graphs of \( f \) and \( h \) compare? Why?

d. Sketch a graph of the function \( p \) given by
   \[ p(x) = f(x + 2). \]
   How do the graphs of \( f \) and \( p \) compare? Why?

e. Example: **Solution** (DOK 3)
   The figure shows the graph of a function \( f \) whose domain is the interval \(-2 \leq x \leq 2\).

![Graph of function f](image)

   a. In (i)-(iii), sketch the graph of the given function and compare with the graph of \( f \). Explain what you see.
      i. \( g(x) = f(x) + 2 \)
      ii. \( h(x) = -f(x) \)
      iii. \( p(x) = f(x + 2) \)

   b. The points labelled \( Q, O, P \) on the graph of \( f \) have coordinates
      \[ Q = (-2, -0.509), \quad O = (0, -0.4), \quad P = (2, 1.309). \]
      What are the coordinates of the points corresponding to \( P, O, Q \) on the graphs of \( g, h, \) and \( p \)?

f. Example: **Solution** (DOK 3)
Let \( f \) be the function defined by \( f(x) = 2x^2 + 4x - 16 \). Let \( g \) be the function defined by

\[
g(x) = 2(x + 1)^2 - 18.
\]

a. Verify that \( f(x) = g(x) \) for all \( x \).

b. In what ways do the equivalent expressions \( 2x^2 + 4x - 16 \) and \( 2(x + 1)^2 - 18 \) help to understand the function \( f \)?

c. Consider the functions \( h, l, m, \) and \( n \) given by

\[
\begin{align*}
h(x) &= x^2 \\
l(x) &= x + 1 \\
m(x) &= x - 9 \\
n(x) &= 2x
\end{align*}
\]

Show that \( f(x) \) is a composition, in some order, of the functions \( h, l, m, \) and \( n \). How do you determine the order of composition?

d. Explain the impact each of the functions \( l, m, \) and \( n \) has on the graph of the composition.

g. Example: Solution (DOK 3)

In this task we will explore the effect that changing the parameters in a sinusoidal function has on the graph of the function. A general sinusoidal function is of the form

\[
y = A \sin(B(x - h)) + k
\]

or

\[
y = A \cos(B(x - h)) + k.
\]

a. Use the sliders in the applet to change the values of \( A, k, h, \) and \( B \) to create the functions in the table. Then describe the effect that changing each parameter has on the shape of the graph. Add more rows to the table, if necessary.
b. Describe how changing $A$, $k$, and $h$ changes the graph of the function.

c. There seems to be a relationship between $B$ and the period of the function but it is harder to describe than the other parameters. Experiment with different values of $B$ and fill in the corresponding period in the table below. In the last row of the table, use the data you have collected to infer a general relationship between $B$ and the period.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$y = \sin(Bx)$</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$y = \sin(x)$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$y = \sin(2x)$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$y = \sin(4x)$</td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>$y = \sin(\frac{1}{2}x)$</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>$y = \sin(-2x)$</td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td>$y = \sin(-4x)$</td>
<td></td>
</tr>
<tr>
<td>-1/2</td>
<td>$y = \sin(-\frac{1}{2}x)$</td>
<td></td>
</tr>
</tbody>
</table>

h. Example: Solution (DOK 3)
The figure shows a graph of the function \( f(x) = x^2 \).

a) For each of the graphs of quadratic functions below, find values of \( a \), \( b \), and \( c \) so that the function \( f(x) = ax^2 + bx + c \) has that graph. (For example, the graph in the first part corresponds to \( a = 1 \), \( b = 0 \), and \( c = 0 \)).

b) For each of the following three descriptions of graphs of quadratic functions, sketch a graph by hand, and then find a function in the form \( f(x) = ax^2 + bx + c \) whose graph fits the description.

- The graph is concave down and has its vertex at \((0, 7)\).
- The graph is concave up and has its vertex at \((-7, 0)\).
- The graph has its vertex at \((4, -9)\) and passes through the point \((6, -1)\).

c) If the graph of the function \( f(x) = ax^2 + bx + c \) is as below, determine the signs of \( a \), \( b \), and \( c \).

i. Example: Solution (DOK 3)
The figure shows a graph of the function $f(x) = x^2$.

a) For each of the graphs of quadratic functions below, find values of $a$, $h$, and $k$ so that the function $f(x) = a(x - h)^2 + k$ has that graph. (For example, the graph in the first part corresponds to $a = 1, h = 0$, and $k = 0$.)
j. Example: The graph of \( y = x^2 \) is shown on the grid. Draw on the grid to show the graph of \( y = (x - 4)^2 + 2 \).
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function $f$ that has an inverse and write an expression for the inverse. For example, $f(x) = 2x^3$ or $f(x) = \frac{x+1}{x-1}$ for $x \neq 1$.
   b. (+) Verify by composition that one function is the inverse of another.
   c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
   d. (+) Produce an invertible function from a non-invertible function by restricting the domain. (F-BF.B.4) (DOK 1,2)
      1. Example: Solution (DOK 2)
The table below shows the number of households in the U.S. in the years 1998-2004 [data source: www.census.gov].

| households (in thousands) | 97,107 | 98,990 | 99,627 | 101,018 | 102,528 | 103,874 | 104,705 |

a. Find a linear function, \( h \), which models the number of households in the U.S. (in thousands) as a function of the year, \( t \).

b. Write an expression for \( h^{-1} \).

c. Find \( h^{-1}(111,000) \) and interpret your answer in terms of the number of households.

2. Example: Solution (DOK 3)

Let \( f \) be the function that assigns to a temperature in degrees Celsius its equivalent in degrees Fahrenheit. Let \( g \) be the function that assigns to a temperature in degrees Kelvin its equivalent in degrees Celsius.

a. Explain what \( x \) and \( f(g(x)) \) represent in terms of temperatures, or explain why there is no reasonable representation.

b. Explain what \( x \) and \( g(f(x)) \) represent in terms of temperatures, or explain why there is no reasonable representation.

c. Given that \( f(x) = \frac{9}{5}x + 32 \) and \( g(x) = x - 273 \), find an expression for \( f(g(x)) \).

d. Find an expression for the function \( h \) which assigns to a temperature in degrees Fahrenheit its equivalent in degrees Kelvin.

3. Example: Solution (DOK 3)

Let \( f \) be the function that assigns to a temperature in degrees Celsius its equivalent in degrees Fahrenheit.

a. The freezing point of water in degrees Celsius is 0 while in degrees Fahrenheit it is 32. The boiling point of water is 100 degrees Celsius and 212 degrees Fahrenheit. Given that the function \( f \) is linear, use this information to find an equation for \( f \).

b. Find the inverse of the function \( f \) and explain its meaning in terms of temperature conversions.

c. Is there a temperature which is the same in degrees Celsius and in degrees Fahrenheit? Explain how you know.

4. Example: Solution (DOK 3)
Let \( f \) be the function defined by \( f(x) = 10^x \) and \( g \) be the function defined by \( g(x) = \log_{10}(x) \).

a. Sketch the graph of \( y = f(g(x)) \). Explain your reasoning.

b. Sketch the graph of \( y = g(f(x)) \). Explain your reasoning.

c. Let \( f \) and \( g \) be any two inverse functions. For which values of \( x \) does \( f(g(x)) = x \)? For which values of \( x \) does \( g(f(x)) = x \)?

5. **Example:** Solution (DOK 2)

Standard maps of the earth are broken into a grid of latitude lines (east-west) and longitude lines (north-south). Consider the function, \( N(\ell) \), the percentage of earth's surface north of a given latitude, \( \ell \) (north of the equator). Several values of \( N(\ell) \) (to the nearest tenth) can be determined using the table below.

<table>
<thead>
<tr>
<th>( \ell ) (degrees)</th>
<th>90</th>
<th>80</th>
<th>70</th>
<th>60</th>
<th>50</th>
<th>40</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>% of surface</td>
<td>0.0</td>
<td>0.8</td>
<td>3.0</td>
<td>6.7</td>
<td>11.7</td>
<td>17.9</td>
<td>25.0</td>
</tr>
</tbody>
</table>

a. Use the data to sketch a graph of \( N(\ell) \) for \( 30 \leq \ell \leq 90 \).

b. Is the graph of \( N(\ell) \) increasing or decreasing?

c. What are the units of \( \ell \)? What are the units of \( N(\ell) \)?

d. What is the value of \( N^{-1}(25) \)?

e. Describe what is meant by the expression, \( N^{-1}(20) \).

6. **Example:** Solution (DOK 3)

The table below shows \( R = f(t) \), the total amount of rain, in centimeters (cm), during a steady rainfall as a function of time, \( t \), in minutes, since the rain started.

<table>
<thead>
<tr>
<th>( t ) (minutes)</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
<th>105</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R ) (cm)</td>
<td>0</td>
<td>0.5</td>
<td>0.75</td>
<td>1.3</td>
<td>2</td>
<td>2.7</td>
<td>3.5</td>
<td>4.2</td>
<td>5</td>
</tr>
</tbody>
</table>

a. Explain why \( f \) is an invertible function.

b. Find \( f(45) \) and interpret it in the context of the situation.

c. Find \( f^{-1}(4.2) \) and interpret it in the context of the situation.

7. **Example:** Solution (DOK 2)
5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents. (F-BF.B.5) (DOK 1,2)
   a. Example: Solution (DOK 3)
      Recall that $\log_b(x)$ is by definition the exponent which $b$ must be raised to in order to yield $x$ ($b > 0$).

      Part I

      a. Use this definition to compute $\log_2(2^5)$.
      b. Use this definition to compute $\log_{10}(0.001)$.
      c. Use this definition to compute $\ln(e^3)$.
      d. Explain why $\log_b(b^y) = y$ where $b > 0$.

      The above technique can be used to raise numbers to logarithmic powers by first simplifying the exponent.

      Part II

      a. Evaluate $10^{\log_{10}(100)}$.
      b. Evaluate $2^{\log_2(\sqrt{2})}$.
      c. Evaluate $e^{\ln(80)}$.
      d. Explain why $b^{\log_b(x)} = x$ where $b > 0$.

   b. Example: Solution (DOK 2)
Below is a picture of the functions \( f(x) = \log_b x \) and \( g(x) = b^x \). In the application below, the base \( b \) varies between 1 and 2 (by hundredths) and pressing the "play" button will run through all possible values of \( b \).

a. For which values of \( b \) do the two graphs appear not to meet?

b. For which values of \( b \) do the two graphs appear to meet in two points?

c. Check by evaluating the functions \( f \) and \( g \) that the two graphs meet at \( x = e \) when \( b = e^{\frac{1}{2}} \).
Construct and compare linear, quadratic, and exponential models and solve problems (F-LE.A)

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.
   b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. \( \text{(F-LE.A.1)} \) \( \text{(DOK 1,2,3)} \)

   1. Example: Solution (DOK 3)

   The following tables show the values of linear, quadratic, and exponential functions at various values of \( x \). Indicate which function type corresponds to each table. Justify your choice.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>3</td>
<td>28</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>4</td>
<td>56</td>
</tr>
</tbody>
</table>

   2. Example: Solution (DOK 3)
The data in the table below was taken from Wikipedia.

<table>
<thead>
<tr>
<th>Year</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1804</td>
<td>1</td>
</tr>
<tr>
<td>1927</td>
<td>2</td>
</tr>
<tr>
<td>1960</td>
<td>3</td>
</tr>
<tr>
<td>1974</td>
<td>4</td>
</tr>
<tr>
<td>1987</td>
<td>5</td>
</tr>
<tr>
<td>1999</td>
<td>6</td>
</tr>
<tr>
<td>2012</td>
<td>7</td>
</tr>
</tbody>
</table>

a. Based on the data in the above table, would a linear function be appropriate to model the relationship between the world population and the year? Explain how you know.

b. Using only the data from 1960 onward in the above table, would a linear function be appropriate to approximate the relationship between the world population and the year? Explain how you know.

c. Based on your work in parts (a) and (b), would a linear function be appropriate to predict the world population in 2200? Explain.

3. Example: **Solution** (DOK 2)

   City Bank pays a simple interest rate of 3% per year, meaning that each year the balance increases by 3% of the initial deposit. National Bank pays an compound interest rate of 2.6% per year, compounded monthly, meaning that each month the balance increases by one twelfth of 2.6% of the previous month's balance.

   a. Which bank will provide the largest balance if you plan to invest $10,000 for 10 years? For 15 years?

   b. Write an expression for \( C(y) \), the City Bank balance, \( y \) years after a deposit is left in the account. Write an expression for \( N(m) \), the National Bank balance, \( m \) months after a deposit is left in the account.

   c. Create a table of values indicating the balances in the two bank accounts from year 1 to year 15. For which years is City Bank a better investment, and for which years is National Bank a better investment?

4. Example: **Solution** (DOK 2)
According to Wikipedia, the International Basketball Federation (FIBA) requires that a basketball bounce to a height of 1300 mm when dropped from a height of 1800 mm.

a. Suppose you drop a basketball and the ratio of each rebound height to the previous rebound height is 1300:1800. Let \( h \) be the function that assigns to \( n \) the rebound height of the ball (in mm) on the \( n^{th} \) bounce. Complete the chart below, rounding to the nearest mm.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( h(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1800</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

b. Write an expression for \( h(n) \).

c. Solve an equation to determine on which bounce the basketball will first have a height of less than 100 mm.

5. **Example:** Solution (DOK 3)

a. Complete the table. In the third column, show your work as demonstrated. What do you notice about the 3rd column?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = 2x + b )</th>
<th>( \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 7 )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>2</td>
<td>( 9 )</td>
<td>( 9 - 7 = 2 )</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Complete the table, showing your work as above. What do you notice about the 3rd column? What is the graphical interpretation of this?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = ax + b )</th>
<th>( \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a \cdot 1 + b )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>2</td>
<td>( a \cdot 2 + b )</td>
<td>( a \cdot 2 + b - (a \cdot 1 + b) = a )</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Let \( y = ax + b \). Let \( x_0 \) be any particular \( x \)-value. Show that if \( x_0 \) is increased by 1, the corresponding \( \Delta y \) is a constant that does not depend on \( x_0 \). What is this constant?

d. Does (a) serve as an example of the result in (c)? Explain.
6. **Example:** Solution (DOK 3)

The figure below shows the graphs of the exponential functions $f(x) = c \cdot 3^x$ and $g(x) = d \cdot 2^x$, for some numbers $c > 0$ and $d > 0$. They intersect at the point $(p, q)$.

![Graph of exponential functions](image)

a. Which is greater, $c$ or $d$? Explain how you know.

b. Imagine you place the tip of your pencil at $(p, q)$ and trace the graph of $g$ out to the point with $x$-coordinate $p + 2$. Imagine I do the same on the graph of $f$. What will be the ratio of the $y$-coordinate of my ending point to the $y$-coordinate of yours?

7. **Example:** Solution (DOK 3)

The below table provides some U.S. Population data from 1982 to 1988:

<table>
<thead>
<tr>
<th>Year</th>
<th>Population (in thousands)</th>
<th>Change in Population (in thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1982</td>
<td>231,664</td>
<td>---</td>
</tr>
<tr>
<td>1983</td>
<td>233,792</td>
<td>233,792 - 231,664 = 2,128</td>
</tr>
<tr>
<td>1984</td>
<td>235,825</td>
<td>2,033</td>
</tr>
<tr>
<td>1985</td>
<td>237,924</td>
<td>2,099</td>
</tr>
<tr>
<td>1986</td>
<td>240,133</td>
<td>2,209</td>
</tr>
<tr>
<td>1987</td>
<td>242,289</td>
<td>2,156</td>
</tr>
<tr>
<td>1988</td>
<td>244,499</td>
<td>2,210</td>
</tr>
</tbody>
</table>

**Notice:** The change in population from 1982 to 1983 is 2,128,000, which is recorded in thousands in the first row of the 3rd column. The other changes are computed similarly. All population numbers in the table are recorded in thousands.

**Source:** [http://www.census.gov/popest/archives/1990s/popclocest.txt](http://www.census.gov/popest/archives/1990s/popclocest.txt)

a. If we were to model the relationship between the U.S. population and the year, would a linear function be appropriate? Explain why or why not.

b. Mike decides to use a linear function to model the relationship. He chooses 2,139, the average of the values in the 3rd column, for the slope. What meaning does this value have in the context of this model?

c. Use Mike's model to predict the U.S. population in 1992.

8. **Example:** Solution (DOK 3)
9. **Example:** [Solution](DOK 3)
a. Complete the table below. Is $\Delta x$ constant? What constant is it? What do you notice about the 3rd column of the table?

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = 3x - 4$</th>
<th>$\Delta y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>----</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2 : (-1) = 3</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Complete the table below. Is $\Delta x$ constant? What constant is it? What do you notice about the 3rd column of the table?

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = 3x - 4$</th>
<th>$\Delta y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>----</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Repeat the construction above the table for the linear equation $y = -2x + 1$. How do your observations in the 3rd column compare to those made for the previous table?

d. Let $y(x) = ax + b$. Let $x_0$ be any particular $x$-value. Show that if $x_0$ is increased by a constant $\Delta x$, the corresponding $\Delta y$ is constant. What is this constant?

e. Is a) an example of the result of d)? Explain.

10. Example: Solution (DOK 3)
a. Complete the table below. Is $\Delta x$ a constant? If so, what constant is it? What do you notice about the 3rd column of the table?

<table>
<thead>
<tr>
<th>x</th>
<th>$f(x) = 108 \cdot (\frac{1}{2})^x$</th>
<th>Successive quotients</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>128</td>
<td>--</td>
</tr>
<tr>
<td>1</td>
<td>64</td>
<td>$\frac{f(1)}{f(0)} = \frac{64}{128} = \frac{1}{2}$</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$\frac{f(2)}{f(1)}$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>--</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>--</td>
</tr>
</tbody>
</table>

b. Complete the table below. Is $\Delta x$ a constant? If so, what constant is it? What do you notice about the 3rd column of the table?

<table>
<thead>
<tr>
<th>x</th>
<th>$f(x) = 108 \cdot (\frac{1}{2})^x$</th>
<th>Successive quotients</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>128</td>
<td>--</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>$\frac{f(2)}{f(0)} = \frac{32}{128} = \frac{1}{4}$</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$\frac{f(4)}{f(2)}$</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>--</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>--</td>
</tr>
</tbody>
</table>

c. Let $f(x) = a \cdot b^x$. Let $x_0$ be any particular $x$-value. Show that if $x_0$ is increased by a constant, $\Delta x$, the successive quotient

$$\frac{f(x_0 + \Delta x)}{f(x_0)}$$

is always the same no matter what $x_0$ is.

d. Is b) an example of the result of c)? Explain.

11. Example: Solution (DOK 3)

The figure below shows the graphs of the functions represented by $f(x) = 4x - a$ and $g(x) = 2x + b$ for some constants $a$ and $b$. They intersect at the point $(p, q)$.

![Graph of functions](image)

a. Label the graph of $f$. Do the same for the graph of $g$.

b. What do $a$ and $b$ represent in the graphs above?

c. Imagine you place the tip of your pencil at point $(p, q)$ and trace the graph of $f$ out to the point with $x$-coordinate $p + 2$. Imagine I do the same on the graph of $g$. How much greater would the $y$-coordinate of your ending point be than mine?

12. Example: Solution (DOK 3)
Carbon 14 is a common form of carbon which decays exponentially over time. The half-life of Carbon 14, that is, the amount of time it takes for half of any amount of Carbon 14 to decay, is approximately 5730 years.

Suppose we have a preserved plant and that the plant, at the time it died, contained 10 micrograms of Carbon 14 (one microgram is equal to one millionth of a gram).

a. Using this information, make a table to calculate how much Carbon 14 remains in the preserved plant after $5730 \times n$ years for $n = 0, 1, 2, 3, 4$.

b. What can you conclude from part (a) about when there is one microgram of Carbon 14 remaining in the preserved plant?

c. How much carbon remains in the preserved plant after $2865 = \frac{5730}{2}$ years? Explain how you know.

d. Using the information from part (c), can you give a more precise response to when there is one microgram of Carbon 14 remaining in the preserved plant?

13. Example: Solution (DOK 3)

The data in the table below was taken from Wikipedia.

<table>
<thead>
<tr>
<th>Year</th>
<th>World Population in Billions (Estimate)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1804</td>
<td>1</td>
</tr>
<tr>
<td>1927</td>
<td>2</td>
</tr>
<tr>
<td>1960</td>
<td>3</td>
</tr>
<tr>
<td>1974</td>
<td>4</td>
</tr>
<tr>
<td>1987</td>
<td>5</td>
</tr>
<tr>
<td>1999</td>
<td>6</td>
</tr>
<tr>
<td>2012</td>
<td>7</td>
</tr>
</tbody>
</table>

a. For each span of years in the table below, assume that the relationship between the population, $P$, and the number of years since the beginning of the time period, $t$, is exponential and then determine the annual rate of growth $r$ for that range of years.

For example, for the range of years 1804 through 1927, we have $P(t) = 1,000,000,000 \cdot b^t$, assume an an exponential relationship. Since 1927 is 123 years after 1804, the population in 1927 can be expressed as $P(123)$ and we have

$$P(123) = 1,000,000,000 \cdot b^{123} = 2,000,000,000.$$  

This means

$$b = \sqrt[123]{2} \approx 1.006$$
or that the population grew at a rate of approximately 1.006 − 1 = 0.006 or 0.6% for each year between 1804 and 1927.

<table>
<thead>
<tr>
<th>Span of Years</th>
<th>Approximate Annual World Population Growth Rate ( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1804 - 1927</td>
<td>0.6%</td>
</tr>
<tr>
<td>1927 - 1960</td>
<td></td>
</tr>
<tr>
<td>1960 - 1974</td>
<td></td>
</tr>
<tr>
<td>1974 - 1987</td>
<td></td>
</tr>
<tr>
<td>1987 - 1999</td>
<td></td>
</tr>
<tr>
<td>1999 - 2012</td>
<td></td>
</tr>
</tbody>
</table>

b. How many times bigger is the growth rate from 1927 to 1960 than the growth rate from 1804 to 1927?

c. Based on your answers to parts (a) and (b) would an exponential function be appropriate to model the relationship between the world population and the year? Explain why or why not.

d. Brainstorm some possible explanations for the overall behavior of the growth rates in part (a).

14. Example: Solution (DOK 2)

Algal blooms routinely threaten the health of the Chesapeake Bay. Phosphates compounds supply a rich source of nutrients for the algae, *Prorocentrum minimum*, responsible for particularly harmful spring blooms known as mahogany tides. These compounds are found in fertilizers used by farmers and find their way into the Bay with run-offs resulting from rainstorms. Favorable conditions result in rapid algae growth ranging anywhere from 0.144 to 2.885 cell divisions per day. Algae concentrations are measured and reported in terms of cells per milliliter (cells/ml). Concentrations in excess of 3,000 cells/ml constitute a bloom.

a. Suppose that heavy spring rains followed by sunny days create conditions that support 1 cell division per day and that prior to the rains *Prorocentrum minimum* concentrations measured just 10 cells/ml. Write an equation for a function that models the relationship between the algae concentration and the number of days since the algae began to divide at the rate of 1 cell division per day.

b. Assuming this rate of cell division is sustained for 10 days, present the resulting algae concentrations over that period in a table. Did these conditions result in a bloom?

c. If conditions support 2 cell divisions per day, when will these conditions result in a bloom?

d. Concentrations in excess of 200,000 cells/ml have been reported in the Bay. Assuming the same conditions as in (c), when will concentrations exceed 200,000 cells/ml?
15. Example: Solution (DOK 3)
A fisherman illegally introduces some fish into a lake, and they quickly propagate. The growth of the population of this new species (within a period of a few years) is modeled by \( P(x) = 5b^x \), where \( x \) is the time in weeks following the introduction and \( b \) is a positive unknown base.

a. Exactly how many fish did the fisherman release into the lake?

b. Find \( b \) if you know the lake contains 33 fish after eight weeks. Show step-by-step work.

c. Instead, now suppose that \( P(x) = 5b^x \) and \( b = 2 \). What is the weekly percent growth rate in this case? What does this mean in every-day language?

16. Example: Solution (DOK 2)
In (a)-(e), say whether the quantity is changing in a linear or exponential fashion.

a. A savings account, which earns no interest, receives a deposit of $723 per month.

b. The value of a machine depreciates by 17% per year.

c. Every week, \( \frac{9}{10} \) of a radioactive substance remains from the beginning of the week.

d. A liter of water evaporates from a swimming pool every day.

e. Every 124 minutes, \( \frac{1}{2} \) of a drug dosage remains in the body.

17. Example: Solution (DOK 2)
SCREEN I
In science class, some students dropped a basketball and allowed it to bounce. They measured and recorded the highest point of each bounce.

![Bouncing Ball](BouncingBall_01.png)

The students' data is shown in the table. The first data point \((n = 0)\) represents the height of the ball the moment the students dropped it.
<table>
<thead>
<tr>
<th>Bounce Number</th>
<th>Measured Height in Inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$h(n)$</td>
</tr>
<tr>
<td>0</td>
<td>233</td>
</tr>
<tr>
<td>1</td>
<td>110</td>
</tr>
<tr>
<td>2</td>
<td>46.6</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
</tr>
</tbody>
</table>

a. Let $n$ be the bounce number and $h(n)$ be the height. Consider the following general forms for different kinds of models where $a$ and $b$ represent numbers:

<table>
<thead>
<tr>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(n) = a \cdot n + b$</td>
<td>$h(n) = a \cdot n^2 + b$</td>
<td>$h(n) = \frac{a}{n} + b$</td>
<td>$h(n) = a \cdot e^{bn}$</td>
</tr>
</tbody>
</table>

Which of the models shown is most appropriate to use for the given data?

[Student could choose to open up a scientific calculator or graph the values given in the table.]

Given the data above, what are reasonable values for $a$ and $b$ if we want to create a specific model to fit the data? Write the appropriate values in the equation below.

[Based on the choice students make above, they are given the appropriate template below. Here is the template for the exponential model]

$h(n) = a \cdot e^{bn}$

b. Based on your model, what will be the first bounce with a maximum height that is less than 1 inch?

Next

[When the student clicks "Next" from this screen, they get a message, "Are you sure you are ready to go on? You cannot change any of your previous answers after you continue from this screen."]

SCREEN II

c. Mika said,

The model I came up with is $h(n) = 233 \cdot e^{-0.8n}$. I used it to predict that after 50 bounces, the height of the bounces will be less than a thousandth of an inch. It is good to have the model because it would be very difficult to measure such small heights.

What is the best way to characterize Mika's claim?

i. Mika's claim is true. The whole point of using models is to make predictions.

ii. Mika's claim is true but she should give a more precise bound for the height of the ball after 50 bounces because the heights will be much much smaller than one thousandth of an inch.
iii. Mika is correct that the model predicts that the bounces will all be less than a thousandth of an inch, but in reality the ball will be at rest before it has bounced 50 times.

iv. Mika is not using the model appropriately. Models can’t be used to make predictions past the given data, only between data points.

v. Mika is not using the model appropriately. The model doesn’t fit the data very well so it can’t be used to make predictions that far in the future.

vi. Mika’s claim is not true. The model states that the ball will be at rest before it gets to 50 bounces so the bounce heights will be zero, which is easy to measure.

Done

18. Example: Solution (DOK 2)

SCREEN I
In science class, some students dropped a basketball and allowed it to bounce. They measured and recorded the highest point of each bounce.

The students’ data is shown in the table and scatterplot. The first data point (n = 0) represents the height of the ball the moment the students dropped it.
In this task, you will choose a function to model the data and use the model to answer some questions.

Next

SCREEN II

a. Compute the first three values in the last column of the table below.

<table>
<thead>
<tr>
<th>Bounce Number</th>
<th>Measured Height in Inches</th>
<th>Factor by which Bounce Height Decreased</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>( h(n) )</td>
<td>( \frac{h(n - 1)}{h(n)} )</td>
</tr>
<tr>
<td>0</td>
<td>233</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>46.6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td></td>
</tr>
</tbody>
</table>

[Student could choose to open up a scientific calculator.]

Back Next

SCREEN III

b. Let \( n \) be the bounce number and \( h(n) \) be the height. Consider the following general forms for different kinds of models where \( a \) and \( b \) represent numbers:

\[
\begin{align*}
\text{Model 1:} & \quad h(n) = a \cdot n + b \\
\text{Model 2:} & \quad h(n) = a \cdot n^2 + b \\
\text{Model 3:} & \quad h(n) = \frac{a}{n} + b \\
\text{Model 4:} & \quad h(n) = a \cdot e^{bn}
\end{align*}
\]

Which of the models shown is most appropriate to use for the given data?  

\( \star \)
Given the data above, what are reasonable values for \( a \) and \( h \) if we want to create a specific model to fit the data? Write the appropriate values in the equation below.

[Based on the choice students make above, they are given the appropriate template below. Here is the template for the exponential model]

\[
h(n) = \boxed{a} e^{\boxed{n}}
\]

c. Based on your model, what will be the first bounce with a maximum height that is less than 1 inch?

Back Next

[When the student clicks “Next” from this screen, they get a message, “Are you sure you are ready to go on? You cannot change any of your previous answers after you continue from this screen.”]

SCREEN IV
d. Mika said,

The model I came up with is \( h(n) = 233 \cdot e^{-0.8n} \). I used it to predict that after 50 bounces, the height of the bounces will be less than a thousandth of an inch. It is good to have the model because it would be very difficult to measure such small heights.

What is the best way to characterize Mika’s claim?

i. Mika’s claim is true. The whole point of using models is to make predictions.

ii. Mika’s claim is true but she should give a more precise bound for the height of the ball after 50 bounces because the heights will be much much smaller than one thousandth of an inch.

iii. Mika is correct that the model predicts that the bounces will all be less than a thousandth of an inch, but in reality the ball will be at rest before it has bounced 50 times.

iv. Mika is not using the model appropriately. Models can’t be used to make predictions past the given data, only between data points.

v. Mika is not using the model appropriately. The model doesn’t fit the data very well so it can’t be used to make predictions that far in the future.

vi. Mika’s claim is not true. The model states that the ball will be at rest before it gets to 50 bounces so the bounce heights will be zero, which is easy to measure.

Done

19. Example: Solution (DOK 3)
Below is a table showing the approximate boiling point of water at different elevations:

<table>
<thead>
<tr>
<th>Elevation (meters above sea level)</th>
<th>Boiling Point (degrees Celsius)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>500</td>
<td>98.2</td>
</tr>
<tr>
<td>1000</td>
<td>96.5</td>
</tr>
<tr>
<td>1500</td>
<td>94.7</td>
</tr>
<tr>
<td>2000</td>
<td>93.1</td>
</tr>
<tr>
<td>2500</td>
<td>91.3</td>
</tr>
</tbody>
</table>

a. Based on the table, if we were to model the relationship between the elevation and the boiling point of water, would a linear function be appropriate? Explain why or why not.

b. Below are some additional values for the boiling point of water at higher elevations:

<table>
<thead>
<tr>
<th>Elevation (meters above sea level)</th>
<th>Boiling Point (degrees Celsius)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5,000</td>
<td>83.2</td>
</tr>
<tr>
<td>6,000</td>
<td>80.3</td>
</tr>
<tr>
<td>7,000</td>
<td>77.2</td>
</tr>
<tr>
<td>8,000</td>
<td>74.3</td>
</tr>
<tr>
<td>9,000</td>
<td>71.5</td>
</tr>
</tbody>
</table>

Based on the table, if we were to model the relationship between the elevation and the boiling point of water, would a linear function be appropriate? Explain why or why not.

c. When the information from both tables is combined, would a linear function be appropriate to model this data? What kind of function would you use to model the data? Why?

20. Example: Solution (DOK 3)
Below are population estimates for the larger metropolitan areas of Paris (France), Shenzhen (China), and Lagos (Nigeria) for each decade between 1950 and 2010:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Paris</td>
<td>6,300,000</td>
<td>7,400,000</td>
<td>8,200,000</td>
<td>8,700,000</td>
<td>9,300,000</td>
<td>9,700,000</td>
<td>10,500,000</td>
</tr>
<tr>
<td>Shenzhen</td>
<td>3100</td>
<td>8000</td>
<td>22,000</td>
<td>58,000</td>
<td>875,000</td>
<td>6,600,000</td>
<td>10,000,000</td>
</tr>
<tr>
<td>Lagos</td>
<td>330,000</td>
<td>760,000</td>
<td>1,400,000</td>
<td>2,600,000</td>
<td>4,800,000</td>
<td>7,300,000</td>
<td>11,000,000</td>
</tr>
</tbody>
</table>

a. For each city, decide if the population data can be accurately modeled by a linear, quadratic, and/or exponential function. Explain.

b. If you found one or more good models for a city population, what predictions would those models make for future decades? Are these reasonable?

21. Example: Solution (DOK 3)

For each or the scenarios below, decide whether the situation can be modeled by a linear function, an exponential function, or neither. For those with a linear or exponential model, create a function which accurately describes the situation.

a. From 1910 until 2010 the growth rate of the United States has been steady at about 1.5% per year. The population in 1910 was about 92,000,000.

b. The circumference of a circle as a function of the radius.

c. According to an old legend, an Indian King played a game of chess with a traveling sage on a beautiful, hand-made chessboard. The sage requested, as reward for winning the game, one grain of rice for the first square, two grains for the second, four grains for the third, and so on for the whole chess board. How many grains of rice would the sage win for the \( n \)th square?

d. The volume of a cube as a function of its side length.

22. Example: Solution (DOK 2)

a. Raphael deposits $10,000 in a bank account which earns 5% interest, compounded annually. If he makes no other deposits or withdrawals, how long will it take for Raphael to have $50,000 in his bank account?

b. The Alpe d'Huez is a famous climb in the Tour de France. The road is about 14 kilometers long. It starts at an elevation of about 740 meters and the elevation gain is about 8% of the distance traveled on the road. What is the approximate elevation after 10 kilometers of the climb? What about at the top?

c. A sample of Radon 222 has a mass of 100 grams. It is radioactive with a half-life of 3.8 days (that is, after every 3.8 days, half of the Radon 222 decays, becoming Polonium 218). Approximately when will 1 gram of the Radon sample remain?
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). *(F-LE.A.2) (DOK 1,2)*

   a. Example: **Solution** (DOK 3)

   Susanna heard some exciting news about a well-known celebrity.

   Within a day she told 4 friends who hadn't heard the news yet.

   By the next day each of those friends told 4 other people who also hadn't yet heard the news.

   By the next day each of those people told four more, and so on.

   a. Assume the rumor continues to spread in this manner. Let \( N \) be the function that assigns to \( d \) the number of people who hear the rumor on the \( d^{th} \) day. Write an expression for \( N(d) \).

   b. On which day will at least 100,000 people hear the rumor for the first time?

   c. How many people will hear the rumor for the first time on the 20th day?

   d. Is the answer to (c) realistic? Explain your reasoning.

   b. Example: **Solution** (DOK 3)

   Albuquerque boasts one of the longest aerial trams in the world. The tram transports people up to Sandia Peak. The table shows the elevation of the tram at various times during a particular ride.

<table>
<thead>
<tr>
<th>Minutes into the Ride</th>
<th>2</th>
<th>5</th>
<th>9</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elevation in Feet</td>
<td>7069</td>
<td>7834</td>
<td>8854</td>
<td>10,129</td>
</tr>
</tbody>
</table>

   a. Write an equation for a function (linear, quadratic, or exponential) that models the relationship between the elevation of the tram and the number of minutes into the ride. Justify your choice.

   b. What was the elevation of the tram at the beginning of the ride?

   c. If the ride took 15 minutes, what was the elevation of the tram at the end of the ride?

   c. Example: **Solution** (DOK 2)
According to Wikipedia, the International Basketball Federation (FIBA) requires that a basketball bounce to a height of 1300 mm when dropped from a height of 1800 mm.

a. Suppose you drop a basketball and the ratio of each rebound height to the previous rebound height is 1300:1800. Let $h_n$ be the function that assigns to $n$ the rebound height of the ball (in mm) on the $n^{th}$ bounce. Complete the chart below, rounding to the nearest mm.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$h(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1800</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

b. Write an expression for $h(n)$.

c. Solve an equation to determine on which bounce the basketball will first have a height of less than 100 mm.

d. Example: Solution (DOK 2)

The graph of a function of the form $f(x) = ab^x$ is shown below. Find the values of $a$ and $b$.

e. Example: Solution (DOK 2)
The graph of a function of the form \( f(x) = ab^x \) is shown below. Find the values of \( a \) and \( b \).

- Example: **Solution** (DOK 3)

  a. Suppose \( P_1 = (0, 5) \) and \( P_2 = (3, -3) \). Sketch \( P_1 \) and \( P_2 \). For which real numbers \( m \) and \( b \) does the graph of a linear function described by the equation \( f(x) = mx + b \) contain \( P_1 \)? Explain. Do any of these graphs also contain \( P_2 \)? Explain.

  b. Suppose \( P_1 = (0, 5) \) and \( P_2 = (0, 7) \). Sketch \( P_1 \) and \( P_2 \). Are there real numbers \( m \) and \( b \) for which the graph of a linear function described by the equation \( f(x) = mx + b \) contains \( P_1 \) and \( P_2 \)? Explain.

  c. Now suppose \( P_1 = (c, d) \) and \( P_2 = (g, h) \) and \( c \) is not equal to \( g \). Show that there is only one real number \( m \) and only one real number \( b \) for which the graph of \( f(x) = mx + b \) contains the points \( P_1 \) and \( P_2 \).

- Example: **Solution** (DOK 2)

  a. Find all quadratic functions described by the equation 
  \[ y = ax^2 + bx + c \]
  whose graph contains the two points \( (1, 0) \) and \( (3, 0) \). How are the graphs of these functions related to one another?

  b. Find all quadratic functions described by the equation 
  \[ y = ax^2 + bx + c \]
  whose graph contains the two points \( (1, 1) \) and \( (3, 7) \). How are the graphs of these functions related to one another?

- Example: **Solution** (DOK 3)
Let $f$ be the function that assigns to a temperature in degrees Celsius its equivalent in degrees Fahrenheit.

a. The freezing point of water in degrees Celsius is 0 while in degrees Fahrenheit it is 32. The boiling point of water is 100 degrees Celsius and 212 degrees Fahrenheit. Given that the function $f$ is linear, use this information to find an equation for $f$.

b. Find the inverse of the function $f$ and explain its meaning in terms of temperature conversions.

c. Is there a temperature which is the same in degrees Celsius and in degrees Fahrenheit? Explain how you know.

i. Example: Solution (DOK 3)

An exponential function is a function of the form $f(x) = ab^x$ where $a$ is a real number and $b$ is a positive real number.

a. Suppose $P = (0, 5)$ and $Q = (3, -3)$. For which real numbers $a$ and $b$ does the graph of the exponential function $f(x) = ab^x$ contain $P$? Explain. Do any of these graphs contain $Q$? Explain.

b. Suppose $R = (2, 0)$. If $f(x) = a \cdot b^x$ is an exponential function whose graph contains $R$ what can you conclude about $a$? What is the graph of $f(x)$ in this case?

j. Example: Solution (DOK 2)
Algal blooms routinely threaten the health of the Chesapeake Bay. Phosphate compounds supply a rich source of nutrients for the algae, *Prorocentrum minimum*, responsible for particularly harmful spring blooms known as mahogany tides. These compounds are found in fertilizers used by farmers and find their way into the Bay with run-offs resulting from rainstorms. Favorable conditions result in rapid algae growth ranging anywhere from 0.144 to 2.885 cell divisions per day. Algae concentrations are measured and reported in terms of cells per milliliter (cells/ml). Concentrations in excess of 3,000 cells/ml constitute a bloom.

a. Suppose that heavy spring rains followed by sunny days create conditions that support 1 cell division per day and that prior to the rains *Prorocentrum minimum* concentrations measured just 10 cells/ml. Write an equation for a function that models the relationship between the algae concentration and the number of days since the algae began to divide at the rate of 1 cell division per day.

b. Assuming this rate of cell division is sustained for 10 days, present the resulting algae concentrations over that period in a table. Did these conditions result in a bloom?

c. If conditions support 2 cell divisions per day, when will these conditions result in a bloom?

d. Concentrations in excess of 200,000 cells/ml have been reported in the Bay. Assuming the same conditions as in (c), when will concentrations exceed 200,000 cells/ml?

k. Example: Solution (DOK 3)

Suppose \( R_1 = (c, d) \) and \( R_2 = (e, f) \) are two different points.

a. Is there always a linear function \( f(x) = mx + b \) whose graph contains the points \( R_1 \) and \( R_2 \)? Explain.

b. Find a linear equation of the form \( ax + by = k \) so that \( (1, 6) \) and \( (4, -3) \) are both solutions to the equation.

c. Find a linear equation of the form \( ax + by = k \) so that \( (2, 8) \) and \( (2, -3) \) are both solutions to the equation.

d. Show that there is always a linear equation of the form \( ax + by = k \) so that \( (c, d) \) and \( (e, f) \) are both solutions to the equation.

I. Example: Solution (DOK 2)
In 1966, a Miami boy smuggled three Giant African Land Snails into the country. His grandmother eventually released them into the garden, and in seven years there were approximately 18,000 of them. The snails are very destructive and had to be eradicated. According to the USDA, it took 10 years and cost $1 million to eradicate them.

a. Assuming the snail population grows exponentially, write an expression for the population, \( P \), in terms of the number, \( t \), of years since their release.

b. How long does it take for the population to double?

c. Assuming the cost of eradicating the snails is proportional to the population, how much would it have cost to eradicate them if
   i. they had started the eradication program a year earlier?
   ii. they had let the population grow unchecked for another year?

m. Example: Solution (DOK 2)

The population of a country is initially 2 million people and is increasing at 4% per year. The country's annual food supply is initially adequate for 4 million people and is increasing at a constant rate adequate for an additional 0.5 million people per year.

a. Based on these assumptions, in approximately what year will this country first experience shortages of food?

b. If the country doubled its initial food supply and maintained a constant rate of increase in the supply adequate for an additional 0.5 million people per year, would shortages still occur? In approximately which year?

c. If the country doubled the rate at which its food supply increases, in addition to doubling its initial food supply, would shortages still occur?

n. Example: Solution (DOK 3)
In order to use Carbon 14 for dating, scientists measure the ratio of Carbon 14 to Carbon 12 in the artifact or remains to be dated. When an organism dies, it ceases to absorb Carbon 14 from the atmosphere and the Carbon 14 within the organism decays exponentially, becoming Nitrogen 14, with a half-life of approximately 5730 years. Carbon 12, however, is stable and so does not decay over time.

Scientists estimate that the ratio of Carbon 14 to Carbon 12 today is approximately 1 to 1,000,000,000,000.

a. Assuming that this ratio has remained constant over time, write an equation for a function which models the ratio of Carbon 14 to Carbon 12 in a preserved plant t years after plant has died.

b. In a particular preserved plant, the ratio of Carbon 14 to Carbon 12 is estimated to be about 1 to 13,000,000,000. What can you conclude about when plant lived? Explain.

c. Dinosaurs are estimated to have lived from about 230,000,000 years ago until about 65,000,000 years ago. Using this information and the given half-life of Carbon 14, explain why this method of dating is not used for dinosaur remains.

Example: Solution (DOK 2)

SCREEN 1

In science class, some students dropped a basketball and allowed it to bounce. They measured and recorded the highest point of each bounce.

The students’ data is shown in the table. The first data point (t = 0) represents the height of the ball the moment the students dropped it.
<table>
<thead>
<tr>
<th>Bounce Number</th>
<th>Measured Height in Inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>( h(n) )</td>
</tr>
<tr>
<td>0</td>
<td>233</td>
</tr>
<tr>
<td>1</td>
<td>110</td>
</tr>
<tr>
<td>2</td>
<td>46.6</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
</tr>
</tbody>
</table>

a. Let \( n \) be the bounce number and \( h(n) \) be the height. Consider the following general forms for different kinds of models where \( a \) and \( b \) represent numbers:

<table>
<thead>
<tr>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(n) = a \cdot n + b )</td>
<td>( h(n) = a \cdot n^2 + b )</td>
<td>( h(n) = \frac{a}{n} + b )</td>
<td>( h(n) = a \cdot e^{bn} )</td>
</tr>
</tbody>
</table>

Which of the models shown is most appropriate to use for the given data?

[Student could choose to open up a scientific calculator or graph the values given in the table.]

Given the data above, what are reasonable values for \( a \) and \( b \) if we want to create a specific model to fit the data? Write the appropriate values in the equation below.

[Based on the choice students make above, they are given the appropriate template below. Here is the template for the exponential model]
b. Based on your model, what will be the first bounce with a maximum height that is less than 1 inch?

Next

When the student clicks "Next" from this screen, they get a message, "Are you sure you are ready to go on? You cannot change any of your previous answers after you continue from this screen."

SCREEN II

Mika said,

The model I came up with is \( h(n) = 233 \cdot e^{-0.8n} \). I used it to predict that after 50 bounces, the height of the bounces will be less than a thousandth of an inch. It is good to have the model because it would be very difficult to measure such small heights.

What is the best way to characterize Mika’s claim?

i. Mika’s claim is true. The whole point of using models is to make predictions.

ii. Mika’s claim is true but she should give a more precise bound for the height of the ball after 50 bounces because the heights will be much much smaller than one thousandth of an inch.

iii. Mika is correct that the model predicts that the bounces will all be less than a thousandth of an inch, but in reality the ball will be at rest before it has bounced 50 times.

iv. Mika is not using the model appropriately. Models can’t be used to make predictions past the given data, only between data points.

v. Mika is not using the model appropriately. The model doesn’t fit the data very well so it can’t be used to make predictions that far in the future.

vi. Mika’s claim is not true. The model states that the ball will be at rest before it gets to 50 bounces so the bounce heights will be zero, which is easy to measure.

Example: Solution (DOK 2)
SCREEN I
In science class, some students dropped a basketball and allowed it to bounce. They measured and recorded the highest point of each bounce.

The students' data is shown in the table and scatterplot. The first data point \((n = 0)\) represents the height of the ball the moment the students dropped it.

<table>
<thead>
<tr>
<th>Bounce Number</th>
<th>Measured Height in Inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>233</td>
</tr>
<tr>
<td>1</td>
<td>110</td>
</tr>
<tr>
<td>2</td>
<td>46.6</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
</tr>
</tbody>
</table>

In this task, you will choose a function to model the data and use the model to answer some questions.

Next

SCREEN II
a. Compute the first three values in the last column of the table below.
SCREEN II
a. Compute the first three values in the last column of the table below.

<table>
<thead>
<tr>
<th>Bounce Number</th>
<th>Measured Height in Inches</th>
<th>Factor by which Bounce Height Decreased</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>233</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>46.6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td></td>
</tr>
</tbody>
</table>

[Student could choose to open up a scientific calculator.]

Back Next

SCREEN III
b. Let n be the bounce number and \( h(n) \) be the height. Consider the following general forms for different kinds of models where \( a \) and \( b \) represent numbers:

<table>
<thead>
<tr>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(n) = an + b )</td>
<td>( h(n) = an^2 + b )</td>
<td>( h(n) = \frac{n}{b} + b )</td>
<td>( h(n) = a \cdot e^{bn} )</td>
</tr>
</tbody>
</table>

Which of the models shown is most appropriate to use for the given data? *

Given the data above, what are reasonable values for \( a \) and \( b \) if we want to create a specific model to fit the data? Write the appropriate values in the equation below.

[Based on the choice students make above, they are given the appropriate template below. Here is the template for the exponential model]

\[ h(n) = ae^{bn} \]

c. Based on your model, what will be the first bounce with a maximum height that is less than 1 inch? *

Back Next

[When the student clicks "Next" from this screen, they get a message, "Are you sure you are ready to go on? You cannot change any of your previous answers after you continue from this screen."]

SCREEN IV
d. Mika said,

*The model I came up with is \( h(n) = 233 \cdot e^{-0.8n} \). I used it to predict that after 50 bounces, the height of the bounces will be less than a thousandth of an inch. It is good to have the model because it would be very difficult to measure such small heights.*

What is the best way to characterize Mika's claim?
i. Mika's claim is true. The whole point of using models is to make predictions.

ii. Mika's claim is true but she should give a more precise bound for the height of the ball after 50 bounces because the heights will be much much smaller than one thousandth of an inch.

iii. Mika is correct that the model predicts that the bounces will all be less than a thousandth of an inch, but in reality the ball will be at rest before it has bounced 50 times.

iv. Mika is not using the model appropriately. Models can't be used to make predictions past the given data, only between data points.

v. Mika is not using the model appropriately. The model doesn't fit the data very well so it can't be used to make predictions that far into the future.

vi. Mika's claim is not true. The model states that the ball will be at rest before it gets to 50 bounces so the bounce heights will be zero, which is easy to measure.

Done

q. Example: Solution (DOK 3)

In 1901, the San Francisco mint produced only 72,664 quarters. By comparison, during other years around the turn of the century they made between 1 million and 2 million quarters. As a result these 1901 San Francisco quarters are extremely rare coins and today, in brand new condition, each one is worth about $60,000.

a. Suppose you put $0.25 in the bank, on the first of January 1901, at 5% interest compounded annually. How much money would you have on January 1, 2013? What if the annual interest rate were 10% or 15%?

b. What can you deduce about the annual appreciation rate of the quarter as a rare coin? Explain.

c. Find the annual appreciation rate of the quarter as a rare coin. Does this agree with your answer to part (b)?

r. Example: Solution (DOK 3)
Below is a table showing the approximate boiling point of water at different elevations:

<table>
<thead>
<tr>
<th>Elevation (meters above sea level)</th>
<th>Boiling Point (degrees Celsius)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>500</td>
<td>98.2</td>
</tr>
<tr>
<td>1000</td>
<td>96.5</td>
</tr>
<tr>
<td>1500</td>
<td>94.7</td>
</tr>
<tr>
<td>2000</td>
<td>93.1</td>
</tr>
<tr>
<td>2500</td>
<td>91.3</td>
</tr>
</tbody>
</table>

a. Based on the table, if we were to model the relationship between the elevation and the boiling point of water, would a linear function be appropriate? Explain why or why not.

b. Below are some additional values for the boiling point of water at higher elevations:

<table>
<thead>
<tr>
<th>Elevation (meters above sea level)</th>
<th>Boiling Point (degrees Celsius)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5,000</td>
<td>83.2</td>
</tr>
<tr>
<td>6,000</td>
<td>80.3</td>
</tr>
<tr>
<td>7,000</td>
<td>77.2</td>
</tr>
<tr>
<td>8,000</td>
<td>74.3</td>
</tr>
<tr>
<td>9,000</td>
<td>71.5</td>
</tr>
</tbody>
</table>

Based on the table, if we were to model the relationship between the elevation and the boiling point of water, would a linear function be appropriate? Explain why or why not.

c. When the information from both tables is combined, would a linear function be appropriate to model this data? What kind of function would you use to model the data? Why?

Example: Solution (DOK 3)
Below are population estimates for the larger metropolitan areas of Paris (France), Shenzhen (China), and Lagos (Nigeria) for each decade between 1950 and 2010:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Paris</td>
<td>6,300,000</td>
<td>7,400,000</td>
<td>8,200,000</td>
<td>8,700,000</td>
<td>9,300,000</td>
<td>9,700,000</td>
<td>10,500,000</td>
</tr>
<tr>
<td>Shenzhen</td>
<td>3100</td>
<td>8000</td>
<td>22,000</td>
<td>58,000</td>
<td>875,000</td>
<td>6,600,000</td>
<td>10,000,000</td>
</tr>
<tr>
<td>Lagos</td>
<td>330,000</td>
<td>760,000</td>
<td>1,400,000</td>
<td>2,600,000</td>
<td>4,800,000</td>
<td>7,300,000</td>
<td>11,000,000</td>
</tr>
</tbody>
</table>

a. For each city, decide if the population data can be accurately modeled by a linear, quadratic, and/or exponential function. Explain.

b. If you found one or more good models for a city population, what predictions would those models make for future decades? Are these reasonable?

t. Example: Solution (DOK 2)

Personal Computer hard disk capacity has grown at a remarkably steady exponential rate for several decades. In 1984 this capacity was about 1/100 of a gigabyte (one gigabyte is 1,000,000,000 bytes). In 1995 the capacity had increased to about 1 gigabyte.

a. Using this information, write an exponential function which models the PC storage capacity by year, starting in 1984.

b. According to your model, about how long does it take for the storage capacity to double?

c. In 2010, the storage capacity had increased to just over 500 gigabytes. Does this agree with the prediction given by your model?

d. According to your model, when will the PC storage capacity reach 100,000 gigabytes (or 100 terabytes)?

u. Example: Solution (DOK 2)
A common claim is that it is impossible to fold a single piece of paper in half more than 7 times (try it!).

Among other attempts, the challenge was taken up on an episode of the TV show Mythbusters, trying to avoid the physical restrictions by beginning with an exceptionally large sheet of paper. Watch the quick summary of their efforts below:

As you can see, the height of the folded sheet of paper increases dramatically as you continue folding. Assuming you started with a large enough sheet of paper, how many folds would it take for the stack of paper to reach the moon?

Example: Solution (DOK 2)
An exponential function is a function of the form $f(x) = ab^x$ for positive real numbers $a$ and $b$.

a. Use the app below to sketch exponential functions for various values of $a$ and $b$. Describe in words the effect of changing $a$ and $b$ on the shape of the graph.

b. Find a function of the form $f(x) = a \cdot b^x$ for each of the four graphs below.
w. Example: Solution (DOK 3)

a. In a carefully controlled biology lab, a population of 100 bacteria reproduces via binary fission. That is, every hour, on the hour, each bacteria splits into two bacteria. Assuming no bacteria deaths, find an expression for the number $P(t)$ of bacteria in the population after $t$ hours.

b. In the next lab over, a population of protists reproduces hourly according to multiple fission. The function which gives the population of protists after $t$ hours is

$$P(t) = 50 \cdot 3^t.$$ 

Interpret the significance of the numbers 50 and 3 in the context of the biological experiment.

x. Example: Solution (DOK 2)
Every day Brian takes 20 mg of a drug that helps with his allergies. His doctor tells him that each hour the amount of drug in his bloodstream decreases by 15%.

a. Construct an exponential function of the form \( f(t) = ab^t \), for constants \( a \) and \( b \), that gives the quantity of the drug, in milligrams, that remains in his bloodstream \( t \) hours after he takes the medication.

b. How much of the drug remains one day after taking it?

c. Do you expect the percentage of the dose that leaves the bloodstream in the first half hour to be more than or less than 15%? What percentage is it?

d. How much of the drug remains one minute after taking it?

y. Example: Solution (DOK 3)

The population of Pittsburgh was about 12,600 in 1820. Between 1820 and 1840 the population grew exponentially, increasing by about 70% each decade.

a. Construct an exponential function in the form \( f(t) = ab^t \) that models the population \( t \) decades after 1820.

b. According to your model, what was the population of Pittsburgh in 1825? What about 1839?

c. According to your model, by what percentage did the population increase between 1826 and 1836?

d. Explain the answer to part (c) in terms of the growth factor \( b \).

z. Example: Solution (DOK 3)
Leo leaves his house one morning and notices a small plant growing in the yard that he has never noticed before. Out of curiosity, he grabs a ruler and measures it; the plant is 3cm tall. Exactly one week later, Leo notices the plant has grown quite a bit so he measures it again; now it is 9cm tall. And one week after that, it measures 27cm.

Leo knows that it is unlikely that the plant will continue to triple in height each week indefinitely, but he starts to wonder about the height of the plant before he started to measure it, and how he could model its growth mathematically. Suppose that the plant follows a rule, “triples in height each week.”

a. Read the information contained in the table to understand what Leo has written so far, and then complete the table. Write any heights that are less than 1 cm as fractions.

<table>
<thead>
<tr>
<th>Week</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>w</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (cm)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>9</td>
<td>27</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Height Expression</td>
<td>$3^3$</td>
<td>$3^3$</td>
<td>$3^3$</td>
<td>$3^3$</td>
<td>$3^3$</td>
<td>$3^3$</td>
<td>$3^3$</td>
<td>$3^3$</td>
<td>$3^3$</td>
<td>$3^3$</td>
<td>$3^3$</td>
</tr>
</tbody>
</table>

i. Express the height of the plant, $h$, as a function of the week it was measured, $w$.

ii. Explain in words the meaning of $h(0)$.

iii. Use your function to find the height of the plant on week -4. Write this value as a fraction. Does the result of the function agree with what you wrote in the table?

b. Suppose that a different plant also demonstrates exponential growth, which means it grows by a constant factor, but the following height measurements are taken instead. (Blank cells are provided as convenient workspace, but you don't necessarily have to write anything in them.)

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>w</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (cm)</td>
<td>10</td>
<td>20</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>Height Expression</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

i. Write a function that expresses the height of this new plant, $g$, of this plant as a function of the week it is measured, $w$.

ii. Use your function to determine the height of the plant, in centimeters, on week -4. Write this value as a fraction.

aa. Example: Solution (DOK 3)
“Exponential decay” describes a situation in which a quantity decreases following a certain pattern. We’re going to investigate a situation that decays exponentially. Read through steps 1-6 below and answer questions a-c before carrying out the investigation.

1. Get your hands on 30 6-sided dice and put them in a container.

2. Find a safe place where you can roll 30 dice at once, like a tray (or the container itself, if it is large enough).

3. Roll the remaining dice (or shake the container thoroughly).

4. Remove any dice that display a "1" and set them aside.

5. Write down how many dice remain.

6. Go back to step 3 and repeat. Stop when you have rolled the dice ten times.

Pre-investigation questions:

a. When you roll 30 dice, how many 1’s do you predict you are likely to roll? Why?

b. After you remove the 1’s on the first roll, what fraction of the original number of dice do you predict will remain?

c. How many times do you predict you might have to roll the dice and remove the 1’s before you get to a roll where you don’t roll any 1’s?
Conduct the Investigation:

Construct a table to record the number of 1’s and the number of dice remaining after each roll, as shown.

<table>
<thead>
<tr>
<th>Roll</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of 1's</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of Dice Remaining</td>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now for some mathematical modeling:

d. Use technology to create a scatter plot with number of dice remaining on the vertical axis and roll number on the horizontal axis. Discuss what you see.

e. Based on what you know about exponential functions and the behavior of dice, explain in words why an exponential model would be appropriate for this situation.

f. Based on what you know about exponential functions and 6-sided dice, write a function of the form \( d(x) = ab^x \) to model the relationship between the roll number, \( x \), and the number of dice remaining, \( d(x) \). Your job here is to decide what numbers to use in place of \( a \) and \( b \). This should not be based on the data you collected -- it is a mathematical model for how dice behave.

g. Graph your function on the same set of axes as your scatter plot. Compare and contrast the graphed model \( d(x) \) to the outcome of the experiment shown in the scatter plot.

h. Suppose someone repeated this activity, except they started with 100 8-sided dice instead of 30 6-sided dice. Write a new function that would model this situation.

Here is a tool that could be used in this task:

bb. Example: SPEEDING TICKETS

New York State wants to change its system for assigning speeding fines to drivers. The current
system allows a judge to assign a fine that is within the ranges shown in Table 1.

### Table 1. New York Speeding Fines

<table>
<thead>
<tr>
<th>Miles per Hour over Speed Limit</th>
<th>Minimum Fine</th>
<th>Maximum Fine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 10</td>
<td>$45</td>
<td>$150</td>
</tr>
<tr>
<td>11 - 30</td>
<td>$90</td>
<td>$300</td>
</tr>
<tr>
<td>31 or more</td>
<td>$180</td>
<td>$600</td>
</tr>
</tbody>
</table>

Some people have complained that the New York speeding fine system is not fair. The New Drivers Association (NDA) is recommending a new speeding fine system. The NDA is studying the Massachusetts system because of claims that it is fairer than the New York system.

### Table 2. Massachusetts Speeding Fines

<table>
<thead>
<tr>
<th>Miles per Hour over Speed Limit</th>
<th>Fine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 10</td>
<td>$100 flat charge</td>
</tr>
<tr>
<td>11 or more</td>
<td>$100 flat charge plus $10 for each additional mph above the first 10 mph</td>
</tr>
</tbody>
</table>

In this task, you will:

- Analyze the speeding fine systems for both New York and Massachusetts.
- Use data to propose a fairer speeding fine system for New York State.

### Part B

Using your model from part A, create an equation to calculate the speeding fine, \( f \), based on the number of miles per hour, \( m \), over the speed limit when \( 1 \leq m \leq 20 \).

This equation will be the start of the proposed new model for the New York speeding fine system.

### Part C

Using your model from part A, create an equation to calculate the speeding fine, \( f \), based on the number of miles per hour, \( m \), over the speed limit when \( m > 20 \).

This equation will complete the proposed new model for the New York speeding fine system.
For this item, a full-credit response (1 point) includes

- writing an equation with a slope ranging between 1 and 3, AND a \( y \)-intercept ranging between 80 and 100 OR
- Writing an equation that matches the (correct or incorrect) line graphed as the first piece of item number 1435.

For example,

- \( f = 2m + 90 \)

For this item, a no-credit response (0 points) includes none of the features of a full-credit response.

For example,

- \( f = 15.5m - 201.5 \)

For this item, a full-credit response (1 point) includes
- Writing an equation with a slope ranging between 13 and 18, AND a \( y \)-intercept ranging between -260 and -120. OR
- Writing an equation that matches the (correct or incorrect) line graphed as the second piece of item number 1435.

For example,

- \( F = 15m - 170 \)

For this item, a no-credit response (0 points) includes both
- All other responses

For example, \( f = 2m + 95 \)

3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. *(F-LE.A.3) (DOK 1,2)*
   a. Example: Solution (DOK 2)
Mr. Wiggins gives his daughter Celia two choices of payment for raking leaves:

i. Two dollars for each bag of leaves filled,

ii. She will be paid for the number of bags of leaves she rakes as follows: two cents for filling one bag, four cents for filling two bags, eight cents for filling three bags, and so on, with the amount doubling for each additional bag filled.

a. If Celia rakes enough to five bags of leaves, should she opt for payment method 1 or 2? What if she fills ten bags of leaves?

b. How many bags of leaves would Celia have to fill before method 2 pays more than method 1?

b. Example: Solution (DOK 3)

The table below shows the values of \(2^x\) and \(2x^3 + 1\) for some whole number values of \(x\):

<table>
<thead>
<tr>
<th>(x)</th>
<th>(2^x)</th>
<th>(2x^3 + 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>55</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>129</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>251</td>
</tr>
</tbody>
</table>

a. The numbers in the third column (values of \(2x^3 + 1\)) are all larger than the numbers in the second column (values of \(2^x\)). Does this remain true if the table is extended to include whole number values up to ten?

b. Explain how you know that the values of \(2^x\) will eventually exceed those of the polynomial \(2x^3 + 1\). What is the smallest whole number value of \(x\) for which this happens?

c. Example: Solution (DOK 3)
Using a scientific calculator, Alex makes the following table listing values of \((1.001)^x\) and \(2x\) for a few inputs:

<table>
<thead>
<tr>
<th>(x)</th>
<th>((1.001)^x)</th>
<th>(2x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.001</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>1.01004512</td>
<td>20</td>
</tr>
<tr>
<td>50</td>
<td>1.05124483</td>
<td>100</td>
</tr>
<tr>
<td>100</td>
<td>1.10511570</td>
<td>200</td>
</tr>
<tr>
<td>500</td>
<td>1.64830942</td>
<td>1000</td>
</tr>
</tbody>
</table>

Alex concludes from the table that the values of \(2x\) grow faster than the values of \((1.001)^x\) so that

\[ 2x > (1.001)^x \]

for all positive values of \(x\). Is Alex correct? Explain how you know.

d. Example: **Solution** (DOK 2)
The population of a country is initially 2 million people and is increasing at 4% per year. The country's annual food supply is initially adequate for 4 million people and is increasing at a constant rate adequate for an additional 0.5 million people per year.

a. Based on these assumptions, in approximately what year will this country first experience shortages of food?

b. If the country doubled its initial food supply and maintained a constant rate of increase in the supply adequate for an additional 0.5 million people per year, would shortages still occur? In approximately which year?

c. If the country doubled the rate at which its food supply increases, in addition to doubling its initial food supply, would shortages still occur?

4. For exponential models, express as a logarithm the solution to \(ab^c = d\) where \(a\), \(c\), and \(d\) are numbers and the base \(b\) is 2, 10, or \(e\); evaluate the logarithm using technology.

**F-LE.A.4** (DOK 1)

a. Example: **Solution** (DOK 2)
Carbon 14 is a common form of carbon which decays over time. The amount of Carbon 14 contained in a preserved plant is modeled by the equation

\[ f(t) = 10e^{-ct}. \]

Time in this equation is measured in years from the moment when the plant dies \((t = 0)\) and the amount of Carbon 14 remaining in the preserved plant is measured in micrograms (a microgram is one millionth of a gram). So when \(t = 0\) the plant contains 10 micrograms of Carbon 14.

a. The half-life of Carbon 14, that is the amount of time it takes for half of the Carbon 14 to decay, is approximately 5730 years. Use this information to find the constant \(c\).

b. If there is currently one microgram of Carbon 14 remaining in the preserved plant, approximately when did the plant die?

b. Example: Solution (DOK 3)

A hospital is conducting a study to see how different environmental conditions influence the growth of streptococcus pneumonia, one of the bacteria which causes pneumonia. Three different populations are studied giving rise to the following equations:

\[ p_1(t) = 1000e^{t/3}, \]
\[ p_2(t) = 1500e^{3t/8}, \]
\[ p_3(t) = 5000e^{t/4}. \]

Here \(t\) represents the number of hours since the beginning of the experiment which lasts for 24 hours and \(p_i(t)\) represents the size of the \(i^{th}\) bacteria population.

a. Explain, in terms of the structure of the expressions defining \(p_1(t)\) and \(p_2(t)\), why these two populations never share the same value at any time during the experiment.

b. Explain, in terms of the structure of the expressions defining \(p_1(t)\) and \(p_3(t)\), why these two populations will be equal at exactly one time during the experiment. Determine this time.

c. Example: Solution (DOK 3)
A cup of hot coffee will, over time, cool down to room temperature. The principle of physics governing the process is Newton's Law of Cooling. Experiments with a covered cup of coffee show that the temperature (in degrees Fahrenheit) of the coffee can be modelled by the following equation

\[ f(t) = 110e^{-0.08t} + 75. \]

Here the time \( t \) is measured in minutes after the coffee was poured into the cup.

a. Explain, using the structure of the expression \( 110e^{-0.08t} + 75 \), why the coffee temperature decreases as time elapses.

b. What is the temperature of the coffee at the beginning of the experiment?

c. After how many minutes is the coffee 140 degrees? After how many minutes is the coffee 100 degrees?

d. Example: **Solution** (DOK 2)

Algal blooms routinely threaten the health of the Chesapeake Bay. Phosphate compounds supply a rich source of nutrients for the algae, *Prorocentrum minimum*, responsible for particularly harmful spring blooms known as mahogany tides. These compounds are found in fertilizers used by farmers and find their way into the Bay with run-offs resulting from rainstorms. Favorable conditions result in rapid algae growth ranging anywhere from 0.144 to 2.885 cell divisions per day. Algae concentrations are measured and reported in terms of cells per milliliter (cells/ml). Concentrations in excess of 3,000 cells/ml constitute a bloom.

a. Suppose that heavy spring rains followed by sunny days create conditions that support 1 cell division per day and that prior to the rains *Prorocentrum minimum* concentrations measured just 10 cells/ml. Write an equation for a function that models the relationship between the algae concentration and the number of days since the algae began to divide at the rate of 1 cell division per day.

b. Assuming this rate of cell division is sustained for 10 days, present the resulting algae concentrations over that period in a table. Did these conditions result in a bloom?

c. If conditions support 2 cell divisions per day, when will these conditions result in a bloom?

d. Concentrations in excess of 200,000 cells/ml have been reported in the Bay. Assuming the same conditions as in (c), when will concentrations exceed 200,000 cells/ml?
e. Example: Solution (DOK 2)

In 1966, a Miami boy smuggled three Giant African Land Snails into the country. His grandmother eventually released them into the garden, and in seven years there were approximately 18,000 of them. The snails are very destructive and had to be eradicated. According to the USDA, it took 10 years and cost $1 million to eradicate them.

a. Assuming the snail population grows exponentially, write an expression for the population, \( P \), in terms of the number, \( t \), of years since their release.

b. How long does it take for the population to double?

c. Assuming the cost of eradicating the snails is proportional to the population, how much would it have cost to eradicate them if
   i. they had started the eradication program a year earlier?
   ii. they had let the population grow unchecked for another year?

f. Example: Solution (DOK 3)

In order to use Carbon 14 for dating, scientists measure the ratio of Carbon 14 to Carbon 12 in the artifact or remains to be dated. When an organism dies, it ceases to absorb Carbon 14 from the atmosphere and the Carbon 14 within the organism decays exponentially, becoming Nitrogen 14, with a half-life of approximately 5730 years. Carbon 12, however, is stable and so does not decay over time.

Scientists estimate that the ratio of Carbon 14 to Carbon 12 today is approximately 1 to 1,000,000,000,000.

a. Assuming that this ratio has remained constant over time, write an equation for a function which models the ratio of Carbon 14 to Carbon 12 in a preserved plant \( t \) years after plant has died.

b. In a particular preserved plant, the ratio of Carbon 14 to Carbon 12 is estimated to be about 1 to 13,000,000,000. What can you conclude about when plant lived? Explain.

c. Dinosaurs are estimated to have lived from about 230,000,000 years ago until about 65,000,000 years ago. Using this information and the given half-life of Carbon 14, explain why this method of dating is not used for dinosaur remains.
Carbon 14 is a form of carbon which decays exponentially over time. The amount of Carbon 14 contained in a preserved plant is modeled by the equation

\[ f(t) = 10 \left( \frac{1}{2} \right)^t. \]

Time in this equation is measured in years from the moment when the plant dies \((t = 0)\) and the amount of Carbon 14 remaining in the preserved plant is measured in micrograms (a microgram is one millionth of a gram). The number \(c\) in the exponential measures the exponential rate of decay of Carbon 14.

a. How many micrograms of Carbon 14 are in the plant at the time it died?

b. The best known estimate for the half-life of Carbon 14, that is the amount of time it takes for half of the Carbon 14 to decay, is 5730 ± 40 years. Use this information to calculate the range of possible values for the constant \(c\) in the equation for \(f\).

c. Use your answer from part (b) to find the range of years when there is one microgram remaining in the preserved plant.

h. Example: Solution (DOK 3)

Graphite is a mineral with many technological uses and it is perhaps most familiar for its use in writing instruments. At the atomic level, it is made of many layers of carbon atoms, each layer arranged in the familiar pattern of hexagonal tiles:

![Graphite pattern](image)

The pattern continues on in all directions and there is a single carbon atom at each vertex.

*Graphene* is a 1 atom thick layer of graphite with many interesting properties and uses. Suppose the thickness of graphene is 200 picometers: one picometer is *one trillionth* of a meter. About how many times would you have to split a 1 mm thick sample of graphite in half in order to get a single layer of graphene? Explain.

i. Example: Solution (DOK 2)
Below is a picture of the functions \( f(x) = \log_b x \) and \( g(x) = b^x \). In the application below, the base \( b \) varies between 1 and 2 (by hundredths) and pressing the "play" button will run through all possible values of \( b \).

a. For which values of \( b \) do the two graphs appear not to meet?

b. For which values of \( b \) do the two graphs appear to meet in two points?

c. Check by evaluating the functions \( f \) and \( g \) that the two graphs meet at \( x = e \) when \( b = e^{1/2} \).

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.
   \( \text{(F-LE.B.5) (DOK 1,2)} \)
   a. Example: Solution (DOK 2)
Lauren keeps records of the distances she travels in a taxi and what she pays:

<table>
<thead>
<tr>
<th>Distance, d, in miles</th>
<th>Fare, F, in dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8.25</td>
</tr>
<tr>
<td>5</td>
<td>12.75</td>
</tr>
<tr>
<td>11</td>
<td>26.25</td>
</tr>
</tbody>
</table>

a. If you graph the ordered pairs \((d, F)\) from the table, they lie on a line. How can you tell this without graphing them?

b. Show that the linear function in part (a) has equation \(F = 2.25d + 1.5\).

c. What do the 2.25 and the 1.5 in the equation represent in terms of taxi rides?

b. Example: Solution (DOK 3)

The below table provides some U.S. Population data from 1982 to 1988:

<table>
<thead>
<tr>
<th>Year</th>
<th>Population (in thousands)</th>
<th>Change in Population (in thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1982</td>
<td>231,664</td>
<td>----</td>
</tr>
<tr>
<td>1983</td>
<td>233,792</td>
<td>233,792 - 231,664 = 2,128</td>
</tr>
<tr>
<td>1984</td>
<td>235,825</td>
<td>2,033</td>
</tr>
<tr>
<td>1985</td>
<td>237,924</td>
<td>2,099</td>
</tr>
<tr>
<td>1986</td>
<td>240,133</td>
<td>2,209</td>
</tr>
<tr>
<td>1987</td>
<td>242,289</td>
<td>2,156</td>
</tr>
<tr>
<td>1988</td>
<td>244,499</td>
<td>2,210</td>
</tr>
</tbody>
</table>

Notice: The change in population from 1982 to 1983 is 2,128,000, which is recorded in thousands in the first row of the 3rd column. The other changes are computed similarly. All population numbers in the table are recorded in thousands.

Source: [http://www.census.gov/popest/archives/1990s/popc00est.txt](http://www.census.gov/popest/archives/1990s/popc00est.txt)

a. If we were to model the relationship between the U.S. population and the year, would a linear function be appropriate? Explain why or why not.

b. Mike decides to use a linear function to model the relationship. He chooses 2,139, the average of the values in the 3rd column, for the slope. What meaning does this value have in the context of this model?
c. Use Mike's model to predict the U.S. population in 1992.

c. Example: Solution (DOK 3)

A cup of hot coffee will, over time, cool down to room temperature. The principle of physics governing the process is Newton's Law of Cooling. Experiments with a covered cup of coffee show that the temperature (in degrees Fahrenheit) of the coffee can be modelled by the following equation

\[ f(t) = 110e^{-0.08t} + 75. \]

Here the time \( t \) is measured in minutes after the coffee was poured into the cup.

a. Explain, using the structure of the expression \( 110e^{-0.08t} + 75 \), why the coffee temperature decreases as time elapses.

b. What is the temperature of the coffee at the beginning of the experiment?

c. After how many minutes is the coffee 140 degrees? After how many minutes is the coffee 100 degrees?

d. Example: Solution (DOK 2)

Suppose a can of cold soda is left in a warm room on a summer day. The graph below shows the temperature of the soda as it gradually increased:

![Graph showing temperature of soda in a room over time.]

The function that describes the temperature, \( F \), of the soda (in degrees Fahrenheit) after \( t \) minutes can be expressed by

\[ F(t) = C - Re^{-kt}, \]

for some positive values of \( C, R, \) and \( k. \)

a. Use the graph to estimate \( C. \)

b. Use the graph to estimate \( R. \)

c. What was the approximate room temperature? What was the initial temperature of the soda when placed in the room?
e. Example: **Solution** (DOK 2)

A fisherman illegally introduces some fish into a lake, and they quickly propagate. The growth of the population of this new species (within a period of a few years) is modeled by \( P(x) = 5b^x \), where \( x \) is the time in weeks following the introduction and \( b \) is a positive unknown base.

a. Exactly how many fish did the fisherman release into the lake?

b. Find \( b \) if you know the lake contains 33 fish after eight weeks. Show step-by-step work.

c. Instead, now suppose that \( P(x) = 5b^x \) and \( b = 2 \). What is the weekly percent growth rate in this case? What does this mean in every-day language?

f. Example: **Solution** (DOK 2)

A preserved plant is estimated to contain 1 microgram (a millionth of a gram) of Carbon 14. The amount of Carbon 14 present in the preserved plant is modeled by the equation

\[
f(t) = A \left( \frac{1}{2} \right)^{\frac{t}{1380}}
\]

where \( t \) denotes time since the death of the plant, measured in years, and \( A \) is the amount of Carbon 14 present in the plant at death, measured in micrograms.

a. How much Carbon 14 was present in the living plant assuming it died 5000 years ago?

b. How much Carbon 14 was present in the living plant assuming it died 10000 years ago?

c. The half-life of Carbon 14 is the amount of time it takes for half of the Carbon 14 to decay. What half-life does the expression for the function \( f \) imply for Carbon 14?

g. Example: **Solution** (DOK 3)
DDT is a toxic agricultural chemical that was used in the United States before it was banned in 1972. DDT has a half-life of 15 years. That means it takes 15 years for one half of a quantity of DDT to degrade into a different, harmless chemical. Suppose an environmental scientist in 2015 measured 9g of DDT in a soil sample taken from land where DDT was once heavily used. The scientist modeled the amount of DDT in the soil, $a$, with the function $a(t) = 9(0.5)^t$. She indicated in her notes that $t$ represented "time."

a. Find $a(0)$. What might this value represent in this context?

b. Find $a(1)$ and $a(-1)$. What might these values represent in this context?

c. Explain, with more specificity, what you think $t$ represents in the function $a(t)$.

**Trigonometric Functions**

Extend the domain of trigonometric functions using the unit circle (F-TF.A)

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle. *(F-TF.A.1) (DOK 1)*
   a. Example: Solution (DOK 2)
The wheels on a bicycle have a radius of 13 inches. One wheel is pictured below:

![Diagram of a wheel with center O and point P on the circle.]

The point $O$ is the center of the wheel and a point $P$ on the circle is chosen. After the wheel has moved forward a distance of $d$, the point $P$ moves to a new point on the circle, marked $Q$ below:

![Diagram showing the movement of point P to Q.]

a. Complete the table below, showing how the angle $QOP$ varies with the distance $d$ that the wheel has traveled forward:

<table>
<thead>
<tr>
<th>$d$</th>
<th>Angle $m(\angle QOP)$ in degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 inch</td>
<td></td>
</tr>
<tr>
<td>10 inches</td>
<td></td>
</tr>
<tr>
<td>3 feet</td>
<td></td>
</tr>
<tr>
<td>7 feet</td>
<td></td>
</tr>
</tbody>
</table>

b. Recall that one radian is the measure of the angle cut out by a one unit length arc on a circle with radius one unit so $m(\angle a) = 1$ radian in the picture below:

![Diagram showing a radian angle.]

More generally, a counterclockwise angle of $\tau$ radians is the angle cut out by an arc of length $\tau$ units on a unit circle. Using this definition of radian angle measure, complete the table below, showing how the angle $QOP$ varies with the distance $d$ that the wheel has traveled forward:
b. Example: **Solution** (DOK 2)

Definition: An angle of 1 radian is defined to be the angle, in the counterclockwise direction, at the center of a unit circle which spans an arc of length 1. The picture below illustrates this definition.

![Diagram of 1 radian]

**Angle measurements in radians**

<table>
<thead>
<tr>
<th>d</th>
<th>m(\angle QOP) in radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 inch</td>
<td></td>
</tr>
<tr>
<td>10 inches</td>
<td></td>
</tr>
<tr>
<td>3 feet</td>
<td></td>
</tr>
<tr>
<td>7 feet</td>
<td></td>
</tr>
</tbody>
</table>

c. What is the relationship between the measure of an angle in degrees and its measure in radians?

2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. *(F-TF.A.2) (DOK 1,2)*
a. Example: Solution (DOK 3)

Below is a picture of a right triangle with \( \alpha \) the measure of angle \( A \):

Joyce knows that the sine of \( \alpha \) is the length of the side opposite \( A \) divided by length of the hypotenuse:

\[
\sin \alpha = \frac{|BC|}{|AC|}.
\]

Joyce says, "the sine of an obtuse angle does not make any sense because I can't make a right triangle with an obtuse angle."

a. Draw a picture and explain how Joyce might define the sine of an obtuse angle.

b. What are \( \sin \frac{3\pi}{4} \) and \( \sin \pi \)? Why?

b. Example: Solution (DOK 3)

a. Sketch graphs of \( f(x) = \cos x \) and \( g(x) = \sin x \).

b. Find a translation of the plane which maps the graph of \( f \) to itself.
Find a reflection of the plane which maps the graph of \( f \) to itself. What trigonometric identities are associated with your translation and reflection?

c. Find a translation of the plane which maps the graph of \( g \) to itself.
Find a reflection of the plane which maps the graph of \( f \) to itself. What trigonometric identities are associated with your translation and reflection?

d. Find a translation of the plane which maps the graph of \( f \) to the graph of \( g \). Find a reflection of the plane which maps the graph of \( f \) to the graph of \( g \). What trigonometric identities are associated with your translation and reflection?

c. Example: Solution (DOK 3)
Below is a picture of an angle $\theta$ in the $x$-$y$ plane with the unit circle sketched in purple:

3. Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for $x$, where $x$ is any real number. (F-TF.A.3) (DOK 1,2)
a. Example: Solution (DOK 1)
Use the unit circle and indicated triangle below to find the exact value of the sine and cosine of the special angle $\pi/4$.

![Unit Circle Diagram]

b. Example: Solution (DOK 2)
In the picture below, the purple circle is the set of points in the plane whose distance from the origin (marked as $A$) is 1, often called the unit circle:

![Unit Circle Diagram]

Also pictured are some other points: $B$ lies on the $x$-axis and the unit circle, $C$ lies on the unit circle, and $D$ is the midpoint of $AB$. Angle $\angle BAC = 60^\circ$.

a. Show that $\triangle ABC$ is equilateral and conclude that $\overrightarrow{CD}$ is perpendicular to $\overrightarrow{AB}$.

b. Use part (a) to find the values of $\sin 60^\circ$ and $\cos 60^\circ$.

c. Consider the following picture where $Q$ and $R$ are on the unit circle and $S$ is the intersection point of $\overrightarrow{QR}$ and the $x$-axis:
Show that \( \triangle PQR \) is equilateral and that \( S \) is the midpoint of \( QR \).

d. Use part (c) to find the values of \( \sin 30^\circ \) and \( \cos 30^\circ \).

c. Example: Solution (DOK 2)

Use the unit circle and indicated triangle below to find the exact value of the sine and cosine of the special angle \( \pi/6 \).

4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions. (F-TF.A.4) (DOK 2)

   a. Example: Solution (DOK 3)
Model periodic phenomena with trigonometric functions (F-TF.B)

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.* (F-TF.B.5) (DOK 1,2)
   a. Example: Solution (DOK 3)
A wheel of radius 0.2 meters begins to move along a flat surface so that the center of the wheel moves forward at a constant speed of 2.4 meters per second. At the moment the wheel begins to turn, a marked point $P$ on the wheel is touching the flat surface.

![Diagram of a wheel with a marked point P and a vector indicating the direction of movement]

a. Write an algebraic expression for the function $y$ that gives the height (in meters) of the point $P$, measured from the flat surface, as a function of $t$, the number of seconds after the wheel begins moving.

b. Sketch a graph of the function $y$ for $t > 0$. What do you notice about the graph? Explain your observations in terms of the real-world context given in this problem.

c. We define the horizontal position of the point $P$ to be the number of meters the point has traveled forward from its starting position, disregarding any vertical movement the point has made. Write an algebraic expression for the function $x$ that gives the horizontal position (in meters) of the point $P$ as a function of $t$, the number of seconds after the wheel begins moving.

d. Sketch a graph of the function $x$ for $t > 0$. Is there a time when the point $P$ is moving backwards? Use your graph to justify your answer.

b. Example: Solution (DOK 3)
Given below is a table that gives the populations of foxes and rabbits in a national park over a 12 month period. Note that each value of $t$ corresponds to the beginning of the month and $t = 0$ corresponds to the beginning of January.

<table>
<thead>
<tr>
<th>$t$, month</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$, number of rabbits</td>
<td>1000</td>
<td>750</td>
<td>567</td>
<td>500</td>
<td>567</td>
<td>750</td>
<td>1000</td>
<td>1250</td>
<td>1433</td>
<td>1500</td>
<td>1433</td>
<td>1250</td>
</tr>
<tr>
<td>$f$, number of foxes</td>
<td>150</td>
<td>143</td>
<td>125</td>
<td>100</td>
<td>75</td>
<td>57</td>
<td>50</td>
<td>57</td>
<td>75</td>
<td>100</td>
<td>125</td>
<td>143</td>
</tr>
</tbody>
</table>

Note that the number of rabbits and the number of foxes are both functions of time.

a. Explain why it is appropriate to model the number of rabbits and foxes as trigonometric functions of time.

b. Find an appropriate trigonometric function that models the number of rabbits, $r(t)$, as a function of time, $t$, in months.

c. Find an appropriate trigonometric function that models the number of foxes, $f(t)$, as a function of time, $t$, in months.

d. Graph both functions and give one possible explanation why one function seems to “chase” the other function.

c. Example: Solution (DOK 3)
Given below are two graphs that show the populations of foxes and rabbits in a national park over a 24 month period.

![Graphs of rabbit and fox populations over 24 months.]

a. Explain why it is appropriate to model the number of rabbits and foxes as trigonometric functions of time.

b. Find an appropriate trigonometric function that models the number of rabbits, \( r(t) \), as a function of time, with \( t \) in months.

c. Find an appropriate trigonometric function that models the number of foxes, \( f(t) \), as a function of time, with \( t \) in months.

d. Example: Solution (DOK 2)
In order to determine how effective solar panels would be at various times of the year it is important to know the amount of daylight at a given location. To be able to make easy calculations, it is often not sufficient to just have tables of data, which are readily available online.

Find an appropriate function that serves as a mathematical model for the hours of daylight in a location of your choice.

Use the model to find the hours of daylight on your birthday.

6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed. (F-TF.B.6) (DOK 1,2)

7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.* (F-TF.B.7) (DOK 1,2,3)

Prove and apply trigonometric identities (F-TF.C)

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle. (F-TF.C.8) (DOK 1,2,3)
   a. Example: Solution (DOK 3)

   ![Diagram of triangle with sides AB, BC, and AC]

   a. In the triangle pictured above show that

   \[
   \left(\frac{|AB|}{|AC|}\right)^2 + \left(\frac{|BC|}{|AC|}\right)^2 = 1
   \]

   b. Deduce that $\sin^2 \theta + \cos^2 \theta = 1$ for any acute angle $\theta$.

   c. If $\theta$ is in the second quadrant and $\sin \theta = \frac{8}{17}$, what can you say about $\cos \theta$? Draw a picture and explain.

   b. Example: Solution (DOK 2)

   Suppose that $\cos \theta = \frac{2}{3}$ and that $\theta$ is in the 4th quadrant. Find $\sin \theta$ and $\tan \theta$ exactly.

   c. Example: Solution (DOK 3)
a. Complete the following table, rounding off each answer to the nearest hundredth if using a calculator. Draw a picture showing the quantities $\sin x$ and $\cos x$ calculated in the table.

<table>
<thead>
<tr>
<th>Angle (degrees)</th>
<th>$\cos x$</th>
<th>$\sin x$</th>
<th>$\cos x + \sin x$</th>
<th>$\cos^2 x + \sin^2 x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>75</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. What patterns do you notice in the second and third columns? Do these patterns hold for other angle measures as well? Explain.

c. What do you notice about the numbers in the fourth column of the table? When does $\cos x + \sin x$ appear to take its maximal value?

d. What do you notice about the numbers in the fifth column of the table? Will this hold for other angle measures? Explain.

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems. *(F-TF.C.9) (DOK 1,2,3)*
   
   a. Example: Solution (DOK 3)
In this task, you will show how all of the sum and difference angle formulas can be derived from a single formula when combined with relations you have already learned.

For the following task, assume that the sum angle formula for sine is true. Namely,

\[
\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi.
\]

a. To derive the difference angle formula for sine, write \(\sin(\theta - \phi)\) as \(\sin(\theta + (-\phi))\) and apply the sum angle formula for sine to the angles \(\theta\) and \(-\phi\). Use the fact that sine is an odd function while cosine is an even function to simplify your answer. Conclude that

\[
\sin(\theta - \phi) = \sin(\theta) \cos(\phi) - \cos(\theta) \sin(\phi).
\]

b. To derive the sum angle formula for cosine, use what you learned in (a) to show that

\[
\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi.
\]

You may want to start with an exploration of \(\sin\left(\frac{\pi}{2} - (\theta + \phi)\right)\).

c. Derive the difference angle formula for cosine,

\[
\cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi.
\]

d. Derive the sum angle formula for tangent,

\[
\tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}.
\]

e. Derive the difference angle formula for tangent,

\[
\tan(\theta - \phi) = \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi}.
\]

b. Example: Solution (DOK 3)

Is there an equilateral triangle \(ABC\) so that \(\overline{AB}\) lies on the \(x\)-axis and \(A\), \(B\), and \(C\) all have integer \(x\) and \(y\) coordinates? Explain.
Mathematics | High School—Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include:

Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.

Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.

Designing the layout of the stalls in a school fair so as to raise as much money as possible.

Analyzing stopping distance for a car.

Modeling savings account balance, bacterial colony growth, or investment growth.

Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.

Analyzing risk in situations such as extreme sports, pandemics, and terrorism.

Relating population statistics to individual predictions.
In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.

In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model— for example, graphs of global temperature and atmospheric CO$_2$ over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.
Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

**Modeling Standards** Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).
Mathematics | High School—Geometry

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of "same shape" and "scale
factor" developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to non-right triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.

Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

**Connections to Equations.** The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.
Geometry Overview

Congruence

- Experiment with transformations in the plane
- Understand congruence in terms of rigid motions
- Prove geometric theorems
- Make geometric constructions

Similarity, Right Triangles, and Trigonometry

- Understand similarity in terms of similarity transformations
- Prove theorems involving similarity
- Define trigonometric ratios and solve problems involving right triangles
- Apply trigonometry to general triangles

Circles

- Understand and apply theorems about circles
- Find arc lengths and areas of sectors of circles

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section
- Use coordinates to prove simple geometric theorems algebraically

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems
- Visualize relationships between two-dimensional and three-dimensional objects

Modeling with Geometry

- Apply geometric concepts in modeling situations
- (IA) Use diagrams consisting of vertices and edges (vertex-edge graphs) to model and solve problems related to networks.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
Congruence G-CO

Experiment with transformations in the plane (G-CO.A)

**Example:** Jose and Tina are studying geometric transformations.

[Diagram of three points and a triangle labeled A, A', and B, with a grid background.]

Jose is able to move triangle $A$ to triangle $A'$ using the following sequence of basic transformations:

1. Reflection across the $x$-axis
2. Reflection across the $y$-axis
3. Translation two units to the right

Tina claims that the same three transformations, done in any order, will always produce the same result.

Explain why Tina’s claim is incorrect.

<table>
<thead>
<tr>
<th>Item</th>
<th>Claim</th>
<th>Domain</th>
<th>Target</th>
<th>DOK</th>
<th>CONTENT</th>
<th>MP</th>
<th>Key</th>
</tr>
</thead>
<tbody>
<tr>
<td>#21</td>
<td>3</td>
<td>G-CO</td>
<td>B</td>
<td>3</td>
<td>HSG.CO.A</td>
<td>2, 3</td>
<td>See Below</td>
</tr>
</tbody>
</table>

**Exemplar:** Tina is incorrect because some orders of basic transformations do not produce the same results. Suppose we move triangle $A$ 2 units to the right first. The point $(4, 3)$ is then $(6, 3)$. Then, we take the reflection across the $x$-axis, which makes that point $(6, -3)$. A reflection of $(6, -3)$ across the $y$-axis gives us $(-6, -3)$, which is not one of the vertices of triangle $A'$. Therefore, the basic transformations done in any order do not produce the same result.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc. *(G-CO.A.1) (DOK 1)*
   a. Example: Solution (DOK 3)
Alex and his friends are studying for a geometry test and one of the main topics covered is parallel lines in a plane. They each write down what they think it means for two distinct lines in a plane to be parallel:

a. Rachel writes, "two distinct lines are parallel when they are both perpendicular to a third line."

b. Alex writes, "two distinct lines are parallel when they do not meet."

c. Briana writes, "two distinct lines are parallel when they have the same slope."

Analyze each definition, indicating if it is mathematically correct and if it has any drawbacks.

b. Example: Solution (DOK 3)

Three students have proposed these ways to describe when two lines \( \ell \) and \( m \) are perpendicular:

a. \( \ell \) and \( m \) are perpendicular if they meet at one point and one of the angles at their point of intersection is a right angle.

b. \( \ell \) and \( m \) are perpendicular if they meet at one point and all four of the angles at their point of intersection are right angles.

c. \( \ell \) and \( m \) are perpendicular if they meet at one point and reflection about \( \ell \) maps \( m \) to itself.

Explain why each of these definitions is correct. What are some of the advantages and disadvantages with each?

2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch). (G-CO.A.2) (DOK 1,2)

a. Example: Solution (DOK 2)
A rigid motion of the plane is a map of the plane to itself which preserves distances between points. Let $f$ be such a function. A point $x$ in the plane is called a fixed point of the rigid motion $f$ if $f(x) = x$.

a. Suppose $f$ is the map which translates $A$ to $B$ where $A$ and $B$ are distinct points in the plane. What are the fixed points of $f$? Explain.

b. Suppose $g$ is the map which rotates the plane by 45 degrees counterclockwise around a point $P$. What are the fixed points of $g$? Explain.

c. Suppose $h$ is the map which reflects the plane about a line $\ell$. What are the fixed points of $h$?

d. Suppose $t$ is any rigid motion of the plane. Explain why there are four possibilities for the set of fixed points of $t$:

- no points
- a single point
- a line
- all points.

For each of the four possibilities, give an example of a transformation whose fixed points match the description.

b. Example: Solution (DOK 3)

Let $f$ be the map which dilates the plane by a factor $r > 0$ with respect to a center $O$. We will denote the image $f(A)$ of a point $A$ by $A'$.

a. Suppose $O, P,$ and $Q$ are collinear. What is the relationship between $|P'Q'|$ and $|PQ|$? Draw a picture and explain.

b. Suppose $O, P,$ and $Q$ are not collinear. What is the relationship between $|P'Q'|$ and $|PQ|$? Draw a picture and explain.

c. Example: Solution (DOK 3)
Suppose \( f \) is the map of the plane which takes each point \((x, y)\) to the point \((x - 3, y)\) and \(g\) is the map of the plane that takes each \((x, y)\) to \((3x, y)\).

a. Below is a triangle \(ABC\).

![Diagram of triangle ABC]

Show the image of \(\triangle ABC\) after applying each of the maps \(f\) and \(g\).

b. Does \(f\) preserve distances and angles? Explain.

c. Does \(g\) preserve distances and angles? Explain.

3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself. \((G\text{-}CO.A.3)\) \((DOK 1,2)\)

a. Example: Solution \((DOK 2)\)

Seven circles of the same size are placed in the pattern shown below:

![Pattern of seven circles]

The six outer circles touch the one in the center and each circle on the outside also touches its two neighbors in the outside ring. Find as many rigid motions of the plane as you can which are symmetries of this configuration of circles.

b. Example: Solution \((DOK 3)\)
Jennifer draws the rectangle $ABCD$ below:

![Diagram of rectangle ABCD]

a. Find all rotations and reflections that carry rectangle $ABCD$ onto itself.

b. Lisa draws a different rectangle and she finds a larger number of symmetries (than Jennifer) for her rectangle. What can you conclude about Lisa's rectangle? Explain.

c. Example: Solution (DOK 3)
There is exactly one reflection and no rotation that sends the convex quadrilateral $ABCD$ onto itself. What shape(s) could quadrilateral $ABCD$ be? Explain.

d. Example: Solution (DOK 2)
Suppose $ABCD$ is a quadrilateral for which there is exactly one rotation, through an angle larger than 0 degrees and less than 360 degrees, which maps it to itself. Further, no reflections map $ABCD$ to itself. What shape is $ABCD$?

e. Example: Solution (DOK 3)
Lisa makes an octagon by successively folding a square piece of paper as follows. First, she folds the square in half vertically and horizontally and also along both diagonals leaving these creases:

Next, Lisa makes four more folds, identifying each pair of adjacent lines of symmetry for the square used for the folds in the first step:

Finally Lisa folds the four corners of her shape along the red creases marked below:

Explain why the shape Lisa has made is a regular octagon.

4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments. (G-CO.A.4) (DOK 2)
   a. Example: Solution (DOK 2)
Consider the following possible definitions for rotation of the plane by an angle $\alpha$ about the point $P$:

a. If $Q$ is a point in the plane, then we send $Q$ to the point $R$ so that the measure of $\angle RPQ$ is $\alpha$.

b. If $Q$ is a point in the plane, then we send $Q$ to $R$, where $|QP| = |RP|$ and the measure of $\angle RPQ$ is $\alpha$.

c. If $Q$ is a point in the plane, then we send $Q$ to $R$, where $|QP| = |RP|$ and the measure of $\angle QPR$ is $\alpha$.

d. If $Q$ is a point in the plane and $C$ is the circle with center $P$ containing $Q$, then we send $Q$ to the point $R$ on $C$ so that the measure of $\angle QPR$ is $\alpha$.

Which, if any, of these definitions are valid (that is, do they make sense and have the desired effect) for all points $P$ and $Q$?

b. Example: Solution (DOK 3)

Carlos finds the following definition of a reflection in a math book:

The reflection $r_\ell$ about a line $\ell$ takes each point $P$ on $\ell$ to itself and takes each point $Q$ not on $\ell$ to the point $r_\ell(Q)$ such that $\ell$ is the perpendicular bisector of $Qr_\ell(Q)$.

Carlos does not find this definition very helpful. He says "the reflection about a line $\ell$ sends each point to its mirror image on the other side of $\ell".

a. In what ways is Carlos' definition of reflection more helpful than one from the math book?

b. In what ways is the math book definition of reflection more helpful than Carlos' definition?

c. Example: Solution (DOK 3)

Suppose $\triangle DEF$ below can be obtained from $\triangle ABC$ by a translation:

![Diagram of triangles]

a. Explain why $ABED$, $ACFD$, and $BCFE$ are parallelograms.

b. Explain why $|AD| = |BE| = |CF|$.

d. Example: Solution (DOK 3)
Below is triangle $ABC$ and a rotated image triangle $DEF$.

5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another. **(G-CO.A.5) (DOK 1,2)**
   a. Example: Solution (DOK 2)

   The triangle in the upper left of the figure below has been reflected across a line into the triangle in the lower right of the figure. Use a straightedge and compass to construct the line across which the triangle was reflected.

   ![Diagram of triangle reflection](image)

   b. Example: Solution (DOK 3)
Triangles $ABC$ and $PQR$ pictured below are congruent:

![Triangles ABC and PQR](image)

a. Show the congruence using rigid motions of the plane.

b. Can the congruence be shown with a single translation, rotation, or reflection? Explain.

c. Is it possible to show the congruence using only translations? Explain.

d. Is it possible to show the congruence using only rotations? Explain.

e. Is it possible to show the congruence using only reflections? Explain.

c. **Example: Solution (DOK 2)**

Suppose $\triangle ABC$ and $\triangle PQR$ are distinct, congruent triangles. Using the steps below, show that a congruence can always be shown with one, two, or three rigid motions:

a. A translation taking $A$ to $P$ (if necessary).

b. A rotation taking $B$, or the image of $B$ after translation, to $Q$ (if necessary).

c. A reflection about $\overrightarrow{PQ}$ which takes $C$, or its image after (a) and (b), to $R$ (if necessary).

d. **Example: Solution (DOK 2)**

a. Sketch graphs of $f(x) = \cos x$ and $g(x) = \sin x$.

b. Find a translation of the plane which maps the graph of $f$ to itself. Find a reflection of the plane which maps the graph of $f$ to itself. What trigonometric identities are associated with your translation and reflection?

c. Find a translation of the plane which maps the graph of $g$ to itself. Find a reflection of the plane which maps the graph of $f$ to itself. What trigonometric identities are associated with your translation and reflection?

d. Find a translation of the plane which maps the graph of $f$ to the graph of $g$. Find a reflection of the plane which maps the graph of $f$ to the graph of $g$. What trigonometric identities are associated with your translation and reflection?
Understand congruence in terms of rigid motions (G-CO.B)

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent. **(G-CO.B.6) (DOK 1,2)**
   a. Example: Solution (DOK 2)
      
      Below is a picture of a regular octagon, which we denote by $O$, and two lines denoted $\ell$ and $m$, each containing one side of the octagon:

      ![Octagon Diagram]

      a. Draw $r_\ell(O)$, the reflection of the octagon about $\ell$.
      b. Draw $r_m(O)$ and $r_m(r_\ell(O))$, the reflections of the two octagons from part (a) about line $m$.
      c. Show that the quadrilateral enclosed by the four octagons $O$, $r_\ell(O)$, $r_m(O)$, and $r_m(r_\ell(O))$ found in parts (a) and (b) is a square.

   b. Example: Solution (DOK 3)
      
      Below is a picture of a regular hexagon, which we denote by $H$, and two lines denoted $\ell$ and $m$, each containing one side of the hexagon:

      ![Hexagon Diagram]

      a. Draw $r_\ell(H)$, the reflection of the hexagon about $\ell$.
      b. Draw $r_m(H)$, the reflection of the hexagon about line $m$, together with $H$ and $r_\ell(H)$.
      c. Show that $H$ and its reflections about the six lines containing its sides make the following pattern:
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. \textit{(G-CO.B.7) (DOK 2,3)}
   a. Example: \textit{Solution} (DOK 3)
      
      Below is a picture of two triangles:

      \[\begin{array}{c}
      \triangle ABC \\
      \triangle DEF
      \end{array}\]

      a. Suppose there is a sequence of rigid motions which maps \(\triangle ABC\) to \(\triangle DEF\). Explain why corresponding sides and angles of these triangles are congruent.

      b. Suppose instead that corresponding sides and angles of \(\triangle ABC\) and \(\triangle DEF\) are congruent. Show that there is a sequence of rigid motions which maps \(\triangle ABC\) to \(\triangle DEF\).

8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. \textit{(G-CO.B.8) (DOK 2,3)}
   a. Example: \textit{Solution} (DOK 3)
In the two triangles below, angle $A$ is congruent to angle $D$, side $AC$ is congruent to side $DF$ and side $AB$ is congruent to side $DE$:

Sally reasons as follows: “If angle $A$ is congruent to angle $D$ then I can move point $A$ to point $D$ so that side $AB$ lies on top of side $DE$ and side $AC$ lies on top of side $DF$. Since $AB$ and $DE$ are congruent as are $AC$ and $DF$ the two triangles match up exactly and so they are congruent.”

Explain Sally’s reasoning for why triangle $ABC$ is congruent to triangle $DEF$ using the language of reflections:

a. Construct a reflection which maps point $A$ to point $D$. Call $B'$ and $C'$ the images of $B$ and $C$ respectively under this reflection.

b. Construct a reflection which does not move $D$ but which sends $B'$ to $E$. Call $C''$ the image of $C'$ under this reflection.

c. Construct a reflection which does not move $D$ or $E$ but which sends $C''$ to $F$. 
b. Example: **Solution** (DOK 3)
   In the picture below segment $AB$ is congruent to segment $DE$, segment $AC$ is congruent to segment $DF$ and segment $BC$ is congruent to segment $EF$:

![Diagram](image)

Show that the two triangles $ABC$ and $DEF$ are congruent via the following steps, which produce a rigid transformation of the plane sending $\triangle ABC$ to $\triangle DEF$.

a. Show that there is a translation of the plane which maps $A$ to $D$. Call $B'$ and $C'$ the images of $B$ and $C$ under this transformation.

b. Show that there is a rotation of the plane which does not move $D$ and which maps $B'$ to $E$. Call $C''$ the image of $C'$ under this transformation.

c. Show that there is a reflection of the plane which does not move $D$ or $E$ and which maps $C''$ to $F$.

d. Example: **Solution** (DOK 3)
   In triangles $ABC$ and $ABD$ below, we are given that angle $BAC$ is congruent to angle $BAD$ and angle $ABC$ is congruent to angle $ABD$. Show that the reflection of the plane about line $AB$ maps triangle $ABD$ to triangle $ABC$.

![Diagram](image)
Josh is told that two triangles $ABC$ and $DEF$ share two sets of congruent sides and one pair of congruent angles: $AB$ is congruent to $DE$, $BC$ congruent to $EF$, and angle $C$ is congruent to angle $F$. He is asked if these two triangles must be congruent. Josh draws the two triangles below and says “They are definitely congruent because they share all three side lengths!”

![Triangle Diagram]

a. Explain Josh’s reasoning using one of the triangle congruence criteria: ASA, SSS, SAS.

b. Give an example of two triangles $ABC$ and $DEF$, fitting the criteria of this problem, which are not congruent.

e. Example: Solution (DOK 3)

Suppose $\triangle ABC$ and $\triangle DEF$ share three corresponding congruent sides as pictured below:

![Triangle Diagram]

Show that $\triangle ABC$ is congruent to $\triangle DEF$ as follows:

a. Apply a translation to move $\triangle ABC$ to $\triangle A'B'C'$ with $A' = D$.

b. Apply a rotation to move $\triangle A'B'C'$ to $\triangle A''B''C''$ with $A'' = D$ and $B'' = E$.

c. Explain why $|A''C''| = |DF|$ and conclude that $\overrightarrow{DE}$ is the perpendicular bisector of $C''D$.

d. Show that reflection over $\overrightarrow{DE}$ maps $F$ to $C''$ and conclude that $\triangle ABC$ is congruent to $\triangle DEF$. 
Prove geometric theorems (G-CO.C)

9. Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment’s endpoints. (G-CO.C.9) (DOK 3)

a. Example: Solution (DOK 3)
Consider a circle with center $O$ and let $P$ be a point on the circle.
Suppose $L$ is a tangent line to the circle at $P$, that is $L$ meets the circle only at $P$.

![Diagram of circle and tangent line]

Show that $OP$ is perpendicular to $L$.

b. Example: Solution (DOK 2)
Suppose $A$ and $B$ are two distinct points in the plane and $L$ is the perpendicular bisector of segment $AB$ as pictured below:

![Diagram of a line segment and its perpendicular bisector]

a. If $C$ is a point on $L$, show that $C$ is equidistant from $A$ and $B$, that is show that $AC$ and $BC$ are congruent.

b. Conversely, show that if $P$ is a point which is equidistant from $A$ and $B$, then $P$ is on $L$.

c. Conclude that the perpendicular bisector of $AB$ is exactly the set of points which are equidistant from $A$ and $B.$
c. Example: Solution (DOK 2)
   In the picture below $l$ and $k$ are parallel:

   ![Parallel Lines Diagram]

   Show that the four angles marked in the picture are congruent.

10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.* (G-CO.C.10) (DOK 3)

   a. Example: Solution (DOK 2)
      Liz places seven pennies in the arrangement shown below:

      ![Penny Arrangement Diagram]

      She notices that each of the six outer pennies touch the one in the center and its two outer neighbors. Liz wonders whether or not this is true if the pennies are replaced with seven nickels or quarters or, more generally, any set of seven equally sized circles.

      What is the answer to Liz's question, that is, is it possible to place six circles around a central circle so that each outer circle touches the one in the center as well as its two outside neighbors?

   b. Example: Solution (DOK 3)
Choose two distinct points \( A \) and \( B \) in the plane.

a. For which points \( C \) is \( \triangle ABC \) a right triangle?

b. For which points \( C \) is \( \triangle ABC \) an obtuse triangle (that is, a triangle with one obtuse angle)?

c. For which points \( C \) is \( \triangle ABC \) an acute triangle (that is, a triangle with three acute angles)?

Justify your responses.

c. Example: \textbf{Solution} (DOK 1)

Below is an equilateral triangle whose side lengths are each 1 unit:

![Equilateral Triangle Diagram]

Find the area of \( \triangle ABC \).

d. Example: \textbf{Solution} (DOK 3)

Suppose \( ABC \) is a triangle. Let \( M \) be the midpoint of \( AB \) and \( P \) the midpoint of \( BC \) as pictured below:

![Triangle with Midpoints Diagram]

\begin{enumerate}
\item Show that \( \overrightarrow{MP} \) and \( \overrightarrow{AC} \) are parallel.
\item Show that \( |AC| = 2|MP| \).
\end{enumerate}

e. Example: \textbf{Solution} (DOK 2)
Below is an isosceles triangle $ABC$ with $|AB| = |AC|:$

![Triangle ABC](image)

Three students propose different arguments for why $m(\angle B) = m(\angle C)$.

a. Ravi says

*If I draw the bisector of $\angle A$ then this is a line of symmetry for $\triangle ABC$ and so $m(\angle B) = m(\angle C).$*

b. Brittney says

*If $M$ is the midpoint of $BC$ then $\triangle ABM$ is congruent to $\triangle ACM$ and so $\angle B$ and $\angle C$ are congruent.*

c. Courtney says

*If $P$ is a point on $BC$ such that $\overrightarrow{AP}$ is perpendicular to $BC$ then $\triangle ABP$ is congruent to $\triangle ACP$ and so $\angle B$ and $\angle C$ are congruent.*

Fill in the details in each argument to show why $m(\angle B) = m(\angle C)$. Can you find another different argument showing that $m(\angle B) = m(\angle C)$?

f. Example: Solution (DOK 3)

Suppose $ABC$ is a triangle in the plane as pictured below:

![Triangle ABC](image)

Suppose $M$ is the midpoint of $AC$ and $N$ is the midpoint of $BC$.

a. Draw the rotation of $\triangle ABC$ by 180 degrees about $M$, labeling the image of $B$ as $B'$.

b. Draw the rotation of $\triangle ABC$ by 180 degrees about $N$, labeling the image of $A$ as $A'$.

c. Explain why $B'$, $C$, and $A'$ are collinear.

d. Deduce that $m(\angle A) + m(\angle B) + m(\angle C) = 180$. 

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11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals. (G-CO.C.11) (DOK 3)*

a. Example: Solution (DOK 3)

Suppose that $ABCD$ is a parallelogram, and that $M$ and $N$ are the midpoints of $AB$ and $CD$, respectively. Prove that $MN = AD$, and that the line $MN$ is parallel to $AD$.

![Diagram of parallelogram with midpoints](image)

b. Example: Solution (DOK 3)

In quadrilateral $ABCD$ pictured below, $AB$ is congruent to $CD$ and $BC$ is congruent to $AD$.

![Diagram of quadrilateral with congruent sides](image)

From the given information, can we deduce that $ABCD$ is a parallelogram? Explain.

c. Example: Solution (DOK 2)
Suppose $AB$ is a line segment and $D$ is a point not on $AB$ as pictured below:

![Diagram](image)

Let $C$ be the point so that $|CD| = |AB|$, $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$, and $ABCD$ is a quadrilateral.

Draw a picture of this situation and show that $ABCD$ is a parallelogram.

d. Example: Solution (DOK 2)

Rhianna has learned the SSS and SAS congruence tests for triangles and she wonders if these tests might work for parallelograms.

a. Suppose $ABCD$ and $EFGH$ are two parallelograms all of whose corresponding sides are congruent, that is $|AB| = |EF|$, $|BC| = |FG|$, $|CD| = |GH|$, and $|DA| = |HE|$. Is it always true that $ABCD$ is congruent to $EFGH$?

b. Suppose $ABCD$ and $EFGH$ are two parallelograms with a pair of congruent corresponding sides, $|AB| = |EF|$ and $|BC| = |FG|$. Suppose also that the included angles are congruent, $m(\angle ABC) = m(\angle EFG)$. Are $ABCD$ and $EFGH$ congruent?

Make geometric constructions (G-CO.D)

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.* (G-CO.D.12) (DOK 2)

   a. Example: Solution (DOK 2)
The triangle in the upper left of the figure below has been reflected across a line into the triangle in the lower right of the figure. Use a straightedge and compass to construct the line across which the triangle was reflected.

b. Example: Solution (DOK 3)
You have been asked to place a warehouse so that it is an equal distance from the three roads indicated on the following map. Find this location and show your work.

a. Show how to fold your paper to physically construct this point as an intersection of two creases.

b. Explain why the above construction works, and in particular why you only needed to make two creases.

c. Example: Solution (DOK 2)
Let $A$ and $B$ be two distinct points in the plane and $AB$ the segment joining them. The goal of this problem is to construct the perpendicular bisector of segment $AB$.

Draw circles with radius $|AB|$ centered at $A$ and $B$ respectively as pictured below:

![Diagram of circles and intersection points P and Q]

The two points of intersection of these circles are labelled $P$ and $Q$. Show that line $PQ$ is the perpendicular bisector of $AB$.

d. Example: Solution (DOK 3)

Suppose $A$ is an angle with vertex $P$, as pictured below:

![Diagram of angle A with vertex P]

a. Draw a circle with center $P$ and with radius $r > 0$. Explain why the circle meets each ray of angle $A$ in a single point. Label these points $Q$ and $R$ respectively.

b. Draw circles with centers $Q$ and $R$ respectively and radius $r$. These circles meet at $P$ and a second point to be labelled $B$. Show that ray $PB$ bisects angle $P$.

e. Example: Solution (DOK 2)
Suppose \( P \) is the vertex of an angle and \( Q \) and \( R \) are points on the two angle rays so that \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \) are congruent:

\[
\begin{array}{c}
Q \\
\downarrow \\
\downarrow \\
\downarrow \\
R
\end{array}
\]

a. If \( M \) is the midpoint of \( \overrightarrow{QR} \) show that \( \overrightarrow{PM} \) bisects \( \angle QPR \).

b. If ray \( \overrightarrow{PS} \) bisects \( \angle QPR \) show that \( \overrightarrow{PS} \) meets \( \overrightarrow{QR} \) at its midpoint.

g. Example: Solution (DOK 3)

Jessica is working to construct an equilateral triangle with origami paper and uses the following steps.

a. First she folds the paper in half and then unfolds it, leaving the crease along the dotted line:

\[
\begin{array}{c}
\vdots
\end{array}
\]

b. Next Jessica folds two corners over to the center crease as shown in the picture below:

\[
\begin{array}{c}
\text{[Diagram of folded corners]}
\end{array}
\]

Explain why the two purple triangles meet in a point as the picture suggests and why the large white triangle is an equilateral triangle.

g. Example: Solution (DOK 3)
Lisa makes an octagon by successively folding a square piece of paper as follows. First, she folds the square in half vertically and horizontally and also along both diagonals leaving these creases:

Next, Lisa makes four more folds, identifying each pair of adjacent lines of symmetry for the square used for the folds in the first step:

Finally Lisa folds the four corners of her shape along the red creases marked below:

Explain why the shape Lisa has made is a regular octagon.

h. Example: Solution (DOK 2)
This task examines the mathematics behind an origami construction of a rectangle whose sides have the ratio \( \sqrt{2} : 1 \). Such a rectangle is called a silver rectangle.

Beginning with a square piece of paper, first fold and unfold it leaving the diagonal crease as shown here:

Next fold the bottom right corner up to the diagonal:

After unfolding then fold the left hand side of the rectangle over to the crease from the previous fold:

Here is a picture, after the last step has been unfolded, with all folds shown and some important points marked. In the picture \( T \) is the reflection of \( S \) about \( \ell \).

a. Suppose \( s \) is the side length of our square. Show that \( |PT| = s \).

b. Show that \( \triangle PQT \) is a 45-45-90 isosceles triangle.

c. Calculate \( |PQ| \) and conclude that \( PQRS \) is a silver rectangle.
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle. (G-CO.D.13) (DOK 2)

a. Example: Solution (DOK 3)
   Let $C$ be a circle with center $O$. Suppose $PR$ and $QS$ are two diameters of $C$ which are perpendicular to one another at $O$ as pictured below:

   ![Diagram of a circle with diameters PR and QS perpendicular at O]

   a. Explain why triangles $POQ$, $QOR$, $ROS$, and $SOP$ are congruent.
   b. Using part (a), deduce that quadrilateral $PQRS$ is a square.
   c. What is the area of $PQRS$? Roughly what percent of the area of $C$ is the area of $PQRS$?

b. Example: Solution (DOK 2)
   Let $C$ be a circle with center $O$ and a diameter meeting $C$ in points $P$ and $S$ as shown below:

   ![Diagram of a circle with a diameter PS]

   a. With straightedge and compass, show how to find a point $Q$ on $C$ so that triangle $OPQ$ is equilateral.
   b. Repeating part (a) show how to find points $R$, $T$, $U$ on $C$ so that $PQRSTU$ is a regular hexagon.
   c. Find the area of $PQRSTU$. How does it compare to the area of $C$?

c. Example: Solution (DOK 3)
Jessica is working to construct an equilateral triangle with origami paper and uses the following steps.

a. First she folds the paper in half and then unfolds it, leaving the crease along the dotted line:

![Dotted Line Diagram]

b. Next Jessica folds two corners over to the center crease as shown in the picture below:

![Folding Diagram]

Explain why the two purple triangles meet in a point as the picture suggests and why the large white triangle is an equilateral triangle.

d. Example: Solution (DOK 2)

Suppose we are given a circle of radius \( r \). The goal of this task is to construct an equilateral triangle whose three vertices lie on the circle. Suppose \( AB \) is a diameter of the circle. Draw a circle with center \( A \) and radius \( r \) and label the two points of intersection of the circles \( P \) and \( Q \) as pictured below:

![Circle Diagram]

a. Show that \( m(\angle ABP) = 30 \) and \( m(\angle ABQ) = 30 \).

b. Show that \( \triangle PBQ \) is an equilateral triangle inscribed in the given circle.
Understand similarity in terms of similarity transformations (G-SRT.A)

Example: The radius of sphere Y is twice the radius of sphere X. A student claims that the volume of sphere Y must be exactly twice the volume of sphere X.

Part A: Write numbers in the boxes to create one example to evaluate the student’s claim.

Part B: Decide whether the student’s claim is true, false, or cannot be determined. Circle the correct option.

<table>
<thead>
<tr>
<th>Item</th>
<th>Claim</th>
<th>Domain</th>
<th>Target</th>
<th>DOK</th>
<th>CONTENT</th>
<th>MP</th>
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<td>A</td>
<td>3</td>
<td>HSG.SRT.A</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

1. Verify experimentally the properties of dilations given by a center and a scale factor:
   a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
   b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor. (G-SRT.A.1) (DOK 2)
1. Example: **Solution** (DOK 3)

Suppose we apply a dilation by a factor of 2, centered at the point P, to the figure below.

![Diagram of a dilation]

a. In the picture, locate the images A', B', and C' of the points A, B, and C under this dilation.

b. Based on your picture in part (a), what do you think happens to the line l when we perform the dilation?

c. Based on your picture in part (a), what appears to be the relationship between the distance A'B' and the distance AB? How about the distances B'C' and BC?

d. Can you prove your observations in part (c)?

2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. (G-SRT.A.2) (DOK 1,2)

   a. Example: **Solution** (DOK 3)
In the picture given below, line segments $AD$ and $BC$ intersect at $X$. Line segments $AB$ and $CD$ are drawn, forming two triangles $AXB$ and $CXD$.

In each part (a)-(d) below, some additional assumptions about the picture are given. In each problem, determine whether the given assumptions are enough to prove that the two triangles are similar; and if so, what the correct correspondence of vertices is. If the two triangles must be similar, prove this result by describing a sequence of similarity transformations that maps one triangle to the other. If not, explain why not.

a. The lengths $AX$ and $XD$ satisfy the equation $2AX = 3XD$.
b. The lengths $AX$, $BX$, $CX$, and $DX$ satisfy the equation $\frac{AX}{BX} = \frac{DX}{CX}$.
c. Lines $AB$ and $CD$ are parallel.
d. Angle $XAB$ is congruent to angle $XCD$.

b. Example: Solution (DOK 3)

Quadrilaterals $ABCD$ and $EFGH$ have three corresponding congruent angles:

- $m(\angle A) = m(\angle E)$
- $m(\angle B) = m(\angle F)$
- $m(\angle C) = m(\angle G)$

a. Must it be true that $m(\angle D) = m(\angle H)$? Explain.

b. If $ABCD$ and $EFGH$ are isosceles trapezoids, are they always similar?

c. If $ABCD$ and $EFGH$ are rhombuses, are they always similar?

c. Example: Solution (DOK 3)
Alicia has two triangles $ABC$ and $PQR$ whose corresponding sides are proportional as pictured below:

Alicia says:

I wonder if they are similar because I don’t have any information about the angles?

What is the answer to Alicia’s question? Explain.

d. Example: Solution (DOK 3)
3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar. \((G\text{-SRT.A.3}) (DOK 2,3)\)
   
a. Example: Solution (DOK 3)
Prove theorems involving similarity (G-SRT.B)

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity. (G-SRT.B.4) (DOK 3)
   a. Example: Solution (DOK 3)
Suppose \( ABC \) is a triangle. Let \( M \) be the midpoint of \( \overline{AB} \) and \( \ell \) the line through \( M \) parallel to \( \overrightarrow{AC} \):

![Diagram of triangle with midpoints and parallel line]

a. Show that angle \( \angle CAB \) is congruent to angle \( \angle PMB \) and that angle \( \angle BPM \) is congruent to angle \( \angle BCA \). Conclude that triangle \( MBP \) is similar to triangle \( ABC \).

b. Use part (a) to show that \( P \) is the midpoint of \( \overline{BC} \).

Example: Solution (DOK 3)

Below is a picture of a right triangle \( ABC \) with right angle \( C \) along with the point \( D \) so that \( \overrightarrow{CD} \) is perpendicular to \( \overrightarrow{AB} \).

![Diagram of right triangle with perpendicular line]

a. Show that \( \triangle ACB \) is similar to \( \triangle ADC \) and to \( \triangle CDB \).

b. Use part (a) to conclude that \( |AC|^2 + |BC|^2 = |AB|^2 \).

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures. (G-SRT.B.5) (DOK 1,2,3)
   a. Example: Solution (DOK 2)
Pablo is practicing bank shots on a standard 4 ft.-by-8 ft. pool table that has a wall on each side, a pocket in each corner, and a pocket at the midpoint of each eight-foot side.

Pablo places the cue ball one foot away from the south wall of the table and one foot away from the west wall, as shown in the diagram below. He wants to bank the cue ball off of the east wall and into the pocket at the midpoint of the north wall.

![Diagram of a pool table with a cue ball placed near the south and west walls and directions for banking the cue ball into the north pocket.]

a. At what point should the cue ball hit the east wall?

b. After Pablo practices banking the cue ball off of the east wall, he tries placing the eight-ball two feet from the east wall, as shown in the diagram below, so that if he shoots the cue ball exactly as he did before, the cue ball will strike the eight-ball directly and sink the eight-ball into the north pocket. How far from the north wall should Pablo place the eight-ball?

![Diagram of a pool table with an eight-ball placed near the east wall and directions for沉 the eight-ball into the north pocket.]

b. Example: Solution (DOK 3)
In the picture below, points $A$ and $B$ are the centers of two circles and they are collinear with point $C$. Also $D$ and $E$ lie on the two respective circles and they are also collinear with point $C$. Finally the two circles touch at a single point on $AB$.

![Diagram of circles and points](image)

What is $|BC|$? Explain.

c. Example: Solution (DOK 2)

Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of $\triangle ABC$?

![Diagram of three squares and line segments](image)

d. Example: Solution (DOK 2)

Point $B$ is due east of point $A$. Point $C$ is due north of point $B$. The distance between points $A$ and $C$ is $10\sqrt{2}$ meters, and $\angle BAC = 45^\circ$. Point $D$ is 20 meters due north of point $C$. The distance $|AD|$ is between which two integers?

e. Example: Solution (DOK 2)

In rectangle $ABCD$, $|AB| = 6$, $|AD| = 30$, and $G$ is the midpoint of $AD$. Segment $AB$ is extended 2 units beyond $B$ to point $E$, and $F$ is the intersection of $ED$ and $BC$. What is the area of $BFDG$?

f. Example: Solution (DOK 3)

Is the quadrilateral with vertices $(-6, 2), (-3, 6), (9, -3), (6, -7)$ a rectangle? Explain.

g. Example: Solution (DOK 2)
Rhianna has learned the SSS and SAS congruence tests for triangles and she wonders if these tests might work for parallelograms.

a. Suppose $ABCD$ and $EFGH$ are two parallelograms all of whose corresponding sides are congruent, that is $|AB| = |EF|$, $|BC| = |FG|$, $|CD| = |GH|$, and $|DA| = |HE|$. Is it always true that $ABCD$ is congruent to $EFGH$?

b. Suppose $ABCD$ and $EFGH$ are two parallelograms with a pair of congruent corresponding sides, $|AB| = |EF|$ and $|BC| = |FG|$. Suppose also that the included angles are congruent, $m(\angle ABC) = m(\angle EFG)$. Are $ABCD$ and $EFGH$ congruent?

Example: Solution (DOK 3)

Suppose we take a square piece of paper and fold it in half vertically and diagonally, leaving the creases shown below:

![Diagram of a square with creases]

Next a fold is made joining the top of the vertical crease to the bottom right corner, leaving the crease shown below: the point $P$ is the intersection of this new crease with the diagonal.

![Diagram of a square with additional crease and point P]

In the diagram below, some additional points are labelled:

![Diagram with additional points labelled]
a. Show that $|AP| = 2|CP|$.

b. Using part (a), explain how to use the point $P$ in order to fold the square into equal thirds.

i. Example: Solution (DOK 3)

Milong and her friends are at the beach looking out onto the ocean on a clear day and they wonder how far away the horizon is.

a. About how far can Milong see out on the ocean?

b. If Milong climbs up onto a lifeguard tower, how far is the horizon in Milong's view?

c. Mount Shishaldin lies on a narrow peninsula in Alaska and is pictured here:

![Image of Mount Shishaldin]

If Milong were to stand atop Mount Shishaldin and look out over the ocean, how far would the horizon be?

d. Based on the answers to the questions above, is the distance of Milong's visual horizon proportional to her elevation above the surface of the earth? Explain.

j. Example: Solution (DOK 2)
Below is a picture of a triangle $ABC$ on the coordinate grid. The red lines are parallel to $\overrightarrow{BC}$:

Suppose $P = (1.2, 1.6)$, $Q = (2, 4)$, and $R = (2.4, 5.2)$. Find the coordinates of the points $u$, $v$, and $w$.

Example: Solution (DOK 3)

Suppose $\ell$ and $k$ are lines with slopes $m$ and $-\frac{1}{m}$ respectively where $m$ is a non-zero real number. The goal of this task is to show that $\ell$ and $k$ are perpendicular. Below is a sample picture of $\ell$ and $k$ along with several marked points.

In this picture, $\overrightarrow{PQ}$ is a horizontal line and $\overrightarrow{OR}$ is a vertical line.

a. Suppose $O$ is chosen as the origin and that $P = (m, -1)$. Find the coordinates of $Q$.

b. Show that $\triangle QRO$ is similar to $\triangle ORP$ with a scale factor of $m$.

c. Deduce that $\ell$ is perpendicular to $k$. 

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Define trigonometric ratios and solve problems involving right triangles (G-SRT.C)

Example: Suppose $A$ is an angle such that $\cos A < \sin A$.
Select all angle measures that are possible values for $A$.

a. $25^\circ$

b. $35^\circ$

c. $45^\circ$

d. $55^\circ$

e. $65^\circ$

f. $75^\circ$

<table>
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<th>Item</th>
<th>Claim</th>
<th>Domain</th>
<th>Target</th>
<th>DOK</th>
<th>CONTENT</th>
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<td>D</td>
<td>2</td>
<td>HSG.SRT.C</td>
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<td></td>
</tr>
</tbody>
</table>

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. (G-SRT.C.6) (DOK 1,2)

a. Example: Solution (DOK 3)
Below is a picture of \( \triangle ABC \):

\[ \begin{array}{c}
\text{C} \\
\text{A} \\
\text{B}
\end{array} \]

a. Draw a triangle \( \triangle DEF \) that is similar (but not congruent) to \( \triangle ABC \).

b. How do \( \frac{|DE|}{|DF|} \) and \( \frac{|AB|}{|AC|} \) compare? Explain.

c. When \( \angle B \) is a right angle, the ratio \( |AB| : |AC| \) is called the cosine of \( \angle A \) while the ratio \( |BC| : |AC| \) is called the sine of \( \angle A \). Why do these ratios depend only on \( \angle A \)?

d. The ratios in part (c) make sense whether or not \( \angle B \) is a right angle but they are only given names (sine and cosine) in this special case. What is special about the case where \( B \) is a right angle?

b. Example: Solution (DOK 3)

a. Below is a picture of a right triangle:

\[ \begin{array}{c}
\text{R} \\
\text{P} \\
\text{Q}
\end{array} \]

In terms of the picture, what are \( \sin \angle P \), \( \cos \angle P \), and \( \tan \angle P \)? Do these values depend on the size of the triangle?

b. Complete the following table, rounding off each answer to the nearest hundredth if using a calculator. Draw a picture showing the meaning of \( \sin x \), \( \cos x \), and \( \tan x \) for an acute angle \( x \).
7. Explain and use the relationship between the sine and cosine of complementary angles. (G-SRT.C.7) (DOK 1,2)
   a. Example: Solution (DOK 3)
      a. Suppose $0^\circ < a < 90^\circ$ is the measure of an acute angle. Draw a picture and explain why $\sin a = \cos (90 - a)$
      b. Are there any angle measures $0^\circ < a < 90^\circ$ for which $\sin a = \cos a$. Explain.
   b. Example: Solution (DOK 3)
      a. For which acute angles $a$ is $\sin a = \cos a$? Explain.
      b. For which acute angles $a$ is $\sin a = \cos 2a$? Explain.

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.* (G-SRT.C.8) (DOK 1,2)
   a. Example: Solution (DOK 3)
Selina has a 9 meter by 12 meter lawn that she wants to water. She has two sprinklers, each of which can water grass within an 8.4-meter radius. Selina wants to set up the two sprinklers so that they are on corners of the lawn. She would like for the sprinklers to water as much of the lawn as possible, because she will have to manually water the part of the lawn that is not covered by the sprinklers.

a. Selina considers two different strategies for placing the sprinklers. One strategy is to put the sprinklers at opposite ends of a 12-meter side of the lawn. The other is to put the sprinklers at opposite corners of the lawn. (See the figures below.) Which strategy appears to be best? Justify your answer.

![Diagram](image)

b. If Selina chooses the better of these two strategies, what percentage of the lawn will she be able to water with the sprinklers?

c. If Selina is allowed to put sprinklers in the interior of her lawn, how many sprinklers does she need to water the entire lawn?

b. Example: Solution (DOK 3)
Seven circles of the same size can be placed in the pattern shown below:

The six outer circles touch the one in the center and each circle on the outside also touches its two neighbors in the outside ring.

a. If twelve circles are placed around a central circle, as pictured below, what is the relationship between the diameters of the outer circles and the diameter of the inner circle? Explain.

b. If four circles are placed around a central circle, as pictured below, what is the relationship between the diameter of the outer circles and the diameter of the inner circle? Explain.

c. Example: Solution (DOK 2)
Below is a table of diameters of different denominations of United States Coins:

<table>
<thead>
<tr>
<th>Denomination</th>
<th>Diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dime</td>
<td>17.9 mm</td>
</tr>
<tr>
<td>Nickel</td>
<td>21.2 mm</td>
</tr>
<tr>
<td>Quarter</td>
<td>24.3 mm</td>
</tr>
<tr>
<td>Half Dollar</td>
<td>30.6 mm</td>
</tr>
</tbody>
</table>

If we place nickels around a central dime, as in the picture below, there is room for five nickels with extra space but not enough room for a sixth nickel:

![Image of five nickels around a central dime]

a. How many dimes fit around a central dime? What about around a central nickel, quarter, or half dollar? Do the dimes fit snugly or is there extra space?

b. How many half dollars fit around a central dime? What about around a central nickel, quarter, or half dollar? Do the half dollars fit snugly or is there extra room?

c. Extending the work in (a) and (b) above, for positive numbers \( r \) and \( s \) how many circles of radius \( s \) will fit around a central circle of radius \( r \)?

d. Example: Solution (DOK 2)
Suppose \( P \) is a point not contained on a line \( L \). Let \( Q \) be the point on \( L \) so that \( PQ \) meets \( L \) perpendicularly and let \( R \) be any other point on \( L \) as in the picture below:

![Diagram of a point \( P \), a point \( Q \) on line \( L \), and a point \( R \) on line \( L \). \( PQ \) meets \( L \) perpendicularly.]

Show that segment \( PQ \) is shorter than segment \( PR \).

e. Example: Solution (DOK 2)

Jerry and Ashley are trying to find out if it is possible to see the lowest point in the USA from the highest point in the USA. It turns out that the highest point in the United States, the peak of Mount Whitney, is only 85 miles from the lowest area, a large flat stretch of land in Death Valley called Badwater Basin.

Jerry thinks that the curvature of the earth will block the view of Badwater Basin from the peak of Mount Whitney. Ashley doesn’t think that this is an issue and that they can neglect the curvature of the earth in their investigation.

As it turns out, it is not possible to see Badwater Basin from Mt. Whitney because the Panamint Range blocks the view. But what if there were no obstacles in the way? More precisely:

\[ \text{Standing on top of Mt. Whitney, is it possible to see a point on the surface of the earth 85 miles away, or would the curvature of the earth prevent this?} \]

f. Example: Solution (DOK 3)

In the July 2013 issue of United Airlines’ Hemisphere Magazine the following article appeared:
Q: When you are flying at 35,000 feet, how far can we see on a clear day? What about at 40,000? At 30,000?

A: There are many factors that affect the visible distance to the horizon, including the curvature of the earth and the effects of atmospheric refraction. To simplify the answer, if we only consider the earth’s curvature, the formula for distance in miles is $1.22 \times \sqrt{\text{height of the aircraft in feet}}$. I won't make you do the math. At 10,000 feet, you can see 122 miles. At 30,000 feet, visibility can be 211 miles; at 40,000 feet, you can see 244 miles. Of course, if you look "above" the horizon, you can see the sun at 93,000,000 miles and light years to the stars.
a. Write down an equation that describes Captain Bowers' method.

b. Create your own mathematical model to find the distance to the horizon from a plane.

c. Compare the two models. What simplifying assumptions did Captain Bowers make to create his model? Are these assumptions justified?

g. Example: Solution (DOK 3)

The goal of this task is to estimate the measure of angles in triangles with integer side lengths.

a. What are the angle measures in a triangle whose three sides each have length of one unit? Explain.

b. What are the approximate measures of the three angles in a triangle whose side lengths are 3, 4, and 5 units respectively? Explain.

c. What are the approximate angle measures of the three angles in a triangle whose side lengths are 2, 3, and 4 units respectively?

h. Example: Consider this right triangle.

![Right Triangle Diagram]

Determine if each expression is equivalent to the length of AC. Select Yes or No for each expression.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
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<tr>
<td>13sin(B)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13cos(A)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12tan(A)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12tan(B)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
i. Example: Consider this right triangle.

![Right Triangle Diagram](image)

Enter the measure of \( \angle CAB \) to the nearest hundredth degree.

<table>
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<td>53.13</td>
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</table>

Apply trigonometry to general triangles (G-SRT.D)

9. (+) Derive the formula \( A = \frac{1}{2} ab \sin(C) \) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side. \((\text{G-SRT.D.9})\) \((\text{DOK 2,3})\)

10. (+) Prove the Laws of Sines and Cosines and use them to solve problems. \((\text{G-SRT.D.10})\) \((\text{DOK 1,2,3})\)

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces). \((\text{G-SRT.D.11})\) \((\text{DOK 1,2})\)
Circles

Understand and apply theorems about circles (G-C.A)

1. Prove that all circles are similar. (G-C.A.1) (DOK 3)
   a. Example: Solution (DOK 3)
      
      For this problem, \((a, b)\) is a point in the \(x-y\) coordinate plane and \(r\) is a positive number.

      a. Using a translation and a dilation, show how to transform the circle with radius \(r\) centered at \((a, b)\) into the circle of radius 1 centered at \((0, 0)\).

      b. Explain how to use your work in part (a) to show that any two circles are similar.

2. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle. (G-C.A.2) (DOK 1,2)
   a. Example: Solution (DOK 2)
      
      Consider a circle with center \(O\) and let \(P\) be a point on the circle.
      Suppose \(L\) is a tangent line to the circle at \(P\), that is \(L\) meets the circle only at \(P\).

      ![Diagram of a circle with a tangent line and a radius]

      Show that \(OP\) is perpendicular to \(L\).

      b. Example: Solution (DOK 2)
Suppose $AB$ is a diameter of a circle and $C$ is a point on the circle different from $A$ and $B$ as in the picture below:

```
\[ \begin{array}{c}
A \\
\quad O \\
B \\
\hline
C \\
\end{array} \]
```

a. Show that triangles $COB$ and $COA$ are both isosceles triangles.

b. Use part (a) and the fact that the sum of the angles in triangle $ABC$ is 180 degrees to show that angle $C$ is a right angle.

c. Example: Solution (DOK 3)

Suppose $ABC$ is a triangle with angle $ACB$ a right angle. Suppose $M$ is the midpoint of segment $AB$ and $P$ the midpoint of segment $BC$ as pictured below:

```
\[ \begin{array}{c}
A \\
M \\
P \\
B \\
\hline
C \\
\end{array} \]
```

a. Show that angle $MPB$ is a right angle.

b. Show that triangle $MPB$ is congruent to triangle $MPC$.

c. Conclude from (b) that triangle $ABC$ is inscribed in the circle with diameter $AB$ and radius $|MA|$.

d. Example: Solution (DOK 2)
Jerry and Ashley are trying to find out if it is possible to see the lowest point in the USA from the highest point in the USA. It turns out that the highest point in the United States, the peak of Mount Whitney, is only 85 miles from the lowest area, a large flat stretch of land in Death Valley called Badwater Basin.

Jerry thinks that the curvature of the earth will block the view of Badwater Basin from the peak of Mount Whitney. Ashley doesn't think that this is an issue and that they can neglect the curvature of the earth in their investigation.

As it turns out, it is not possible to see Badwater Basin from Mt. Whitney because the Panamint Range blocks the view. But what if there were no obstacles in the way? More precisely:

Standing on top of Mt. Whitney, is it possible to see a point on the surface of the earth 85 miles away, or would the curvature of the earth prevent this?

3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle. *(G-C.A.3) (DOK 2,3)*
   a. Example: Solution (DOK 3)

   You have been asked to place a warehouse so that it is an equal distance from the three roads indicated on the following map. Find this location and show your work.

   ![Map with roads and points A, B, C, and D labeled.]

   a. Show how to fold your paper to physically construct this point as an intersection of two creases.

   b. Explain why the above construction works, and in particular why you only needed to make two creases.

   b. Example: Solution (DOK 3)
You have been asked to place a fire hydrant so that it is an equal distance from three locations indicated on the following map.

```
A

B

C
```

a. Show how to fold your paper to physically construct this point as an intersection of two creases.

b. Explain why the above construction works, and in particular why you only needed to make two creases.

c. Example: Solution (DOK 2)

The goal of this task is to construct the **circumcenter** of a triangle, that is, the point which is the same distance from each of the triangle's three vertices. A sample triangle is pictured below:

```
A

B

C
```

The collection of points equidistant from $B$ and $C$ is the perpendicular bisector $l$ of $BC$, pictured below:

```
A

B

C
```

The collection of points equidistant from $A$ and $C$ is the perpendicular bisector $m$ of $AC$, and we let $O$ be the point of intersection of $l$ and $m$, displayed below:
Show that \(O\) also lies on the perpendicular bisector of \(AB\).

d. Example: **Solution** (DOK 2)
The circumcenter of a triangle \(ABC\) is the point in the plane which is equidistant from the three vertices \(A, B,\) and \(C\).

a. If \(O\) is the circumcenter of \(\triangle ABC\), show that \(\triangle ABC\) can be inscribed in a circle with center \(O\).

b. Show that \(\triangle ABC\) cannot have more than one circumcenter.

e. Example: **Solution** (DOK 3)
The goal of this task is to show how to draw a circle which is tangent to all three sides of a given triangle: that is, the circle touches each side of the triangle in a single point. Suppose \(ABC\) is a triangle as pictured below with ray \(AO\) the bisector of angle \(A\) and ray \(BO\) the bisector of angle \(B\):

Also pictured are the point \(M\) on \(AC\) so that \(OM\) meets \(AC\) in a right angle and similarly point \(N\) on \(AB\) is chosen so that \(ON\) meets \(AB\) in a right angle.

a. Show that triangle \(AOM\) is congruent to triangle \(AON\).

b. Show that \(OM\) is congruent to segment \(ON\).

c. Arguing as in parts (a) and (b) show that if \(P\) is the point on \(BC\) so that \(OP\) meets \(BC\) in a right angle then \(OP\) is also congruent to \(OM\).

d. Show that the circle with center \(O\) and radius \(|OM|\) is inscribed inside triangle \(ABC\).
f. Example: Solution (DOK 2)
Suppose a circle with center \( O \) is inscribed inside triangle \( ABC \) as in the picture below:

Also pictured above are the three radii of the inscribed circle which meet the triangle at the three points \( P, Q, \) and \( R. \)

a. Show that triangle \( BPO \) is congruent to triangle \( BQO. \)

b. Show that ray \( \overrightarrow{BO} \) is the bisector of angle \( B. \)

c. Show that rays \( \overrightarrow{AO} \) and \( \overrightarrow{CO} \) bisect the respective angles \( A \) and \( C. \)

d. Show that if a circle is inscribed in a triangle then the center of that circle is contained on all three angle bisectors for the angles of the triangle.

g. Example: Solution (DOK 3)
A cyclic quadrilateral is a quadrilateral whose vertices all lie on a circle. An example is pictured below:

Prove that the opposite angles in a cyclic quadrilateral that contains the center of the circle are supplementary.

h. Example: Solution (DOK 3)
Below is a picture of a triangle circumscribed about a circle:

Suppose we are given a circle \( X \) and 3 points on the circle, \( P, Q, \) and \( R \). When is there a triangle \( ABC \) circumscribed about \( X \), tangent to \( X \) at \( P, Q, \) and \( R \)? Explain.

4. (+) Construct a tangent line from a point outside a given circle to the circle. \( \textbf{(G-C.A.4) (DOK 2)} \)
   a. Example: Solution (DOK 2)
      Suppose \( C \) is a circle with center \( O \) and \( P \) is a point outside of \( C \). Let \( M \) be the midpoint of segment \( OP \) and let \( D \) be the circle with center \( M \) passing through \( O \).

Let \( A \) and \( B \) be the two points of intersection of \( C \) and \( D \), pictured below along with several line segments of interest:

a. Show that angles \( OAP \) and \( OBP \) are right angles.

b. Show that \( PA \) and \( PB \) are tangent lines from \( P \) to the circle \( C \).
Find arc lengths and areas of sectors of circles (G-C.B)

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. (G-C.B.5) (DOK 1,2,3)
a. Example: Solution (DOK 2)
   Three circles, each having radius 2, are mutually tangent as pictured below:

What is the total area of the circles together with the shaded region?
Translate between the geometric description and the equation for a conic section (G-GPE.A)

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation. (G-GPE.A.1) (DOK 1,2,3)
   a. Example: Solution (DOK 2)
      Suppose \( A = (a_1, a_2) \) and \( B = (b_1, b_2) \) are two points in the plane, determined by constants \( a_1, a_2, b_1, b_2 \). Suppose \( X = (x_1, x_2) \) is a third point, determined by the variables \( x_1 \) and \( x_2 \).

      a. Write an expression that gives the slope of line \( AX \) in terms of \( a_1, a_2, x_1, x_2 \). Write an expression that gives the slope of line \( BX \) in terms of \( b_1, b_2, x_1, x_2 \).

      b. Write a polynomial equation involving \( a_1, a_2, b_1, b_2, x_1, x_2 \) that expresses that the lines \( AX \) and \( BX \) are perpendicular.

      c. What geometric figure is the solution set of the equation in b)?
   b. Example: Solution (DOK 3)
      This problem examines equations defining different circles in the \( x\)-\( y \) plane.

      a. Use the Pythagorean theorem to find an equation in \( x \) and \( y \) whose solutions are the points on the circle of radius 2 with center \((1, 1)\) and explain why it works.

      b. Suppose \( r \) is a positive number and \((a, b)\) a point in the plane. Use the Pythagorean theorem to find an equation in \( x \) and \( y \) whose solutions are the points on the circle of radius \( r \) with center \((a, b)\) and explain why it works.

2. Derive the equation of a parabola given a focus and directrix. (G-GPE.A.2) (DOK 1,2)
   a. Example: Solution (DOK 2)
3. Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant. \((G\text{-GPE.A.3})\) \((DOK 1, 2)\)

4. Use coordinates to prove simple geometric theorems algebraically. \((G\text{-GPE.B})\)
   a. Example: \textbf{Solution} \((DOK 3)\)
      Draw a quadrilateral \(ABCD\). Try to draw your quadrilateral so that no two sides are congruent, no two angles are congruent, and no two sides are parallel.
      a. Let \(P, Q, R, \text{ and } S\) be the midpoints of sides \(AB, BC, CD, \text{ and } DA\), respectively. Use a ruler to locate these points as precisely as you can, and join them to form a new quadrilateral \(PQRS\). What do you notice about the quadrilateral \(PQRS\)?
      b. Suppose your quadrilateral \(ABCD\) lies in the coordinate plane. Let \((x_1, y_1)\) be the coordinates of vertex \(A\), \((x_2, y_2)\) the coordinates of \(B\), \((x_3, y_3)\) the coordinates of \(C\), and \((x_4, y_4)\) the coordinates of \(D\). Use coordinates to prove the observation you made in part (a).
b. Example: Solution (DOK 2)

Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of \( \triangle ABC \)?

5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point). (G-GPE.B.5) (DOK 1,2)

a. Example: Solution (DOK 3)

Draw a quadrilateral \( ABCD \). Try to draw your quadrilateral so that no two sides are congruent, no two angles are congruent, and no two sides are parallel.

a. Let \( P, Q, R, \) and \( S \) be the midpoints of sides \( AB, BC, CD, \) and \( DA \), respectively. Use a ruler to locate these points as precisely as you can, and join them to form a new quadrilateral \( PQRS \). What do you notice about the quadrilateral \( PQRS \)?

b. Suppose your quadrilateral \( ABCD \) lies in the coordinate plane. Let \( (x_1, y_1) \) be the coordinates of vertex \( A \), \( (x_2, y_2) \) the coordinates of \( B \), \( (x_3, y_3) \) the coordinates of \( C \), and \( (x_4, y_4) \) the coordinates of \( D \). Use coordinates to prove the observation you made in part (a).

b. Example: Solution (DOK 2)

Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of \( \triangle ABC \)?

c. Example: Solution (DOK 2)
Suppose \( A = (-1, 0) \) and \( B = (1, 0) \) are points in the coordinate plane as pictured below:

Suppose \( C = (x, y) \) is a third point in the plane, not on the \( x \)-axis. Show that \( \angle ACB \) is a right angle if and only if \( (x, y) \) lies on the circle \( x^2 + y^2 = 1 \).

d. Example: Solution (DOK 2)
   
   a. On graph paper, sketch a line segment with end points \( A = (0, 2) \) and \( B = (0, 6) \). Plot all points \( C = (x, y) \) such that the triangle \( ABC \) has an area of 6 square units.
   
   b. What two basic geometric theorems are needed to solve this problem?

e. Example: Solution (DOK 2)
   
   a. Given a line segment with end points \( A = (0, 0) \) and \( B = (6, 8) \), find all points \( C = (x, y) \) such that the triangle with vertices \( A, B, C \) has an area of 20 square units.
   
   b. What mathematical results did you apply to solve this problem?

f. Example: Solution (DOK 3)
Suppose \( \ell \) and \( m \) are lines containing \((0, 0)\) as pictured below:

![Diagram of lines \( \ell \) and \( m \) intersecting at \((0, 0)\) and a unit circle]

The unit circle is also shown in the picture as well as two points where the two lines intersect the circle.

a. If \( \ell \) and \( m \) are perpendicular, what is the relationship between the coordinates of the points labeled \( P \) and \( Q \)?

b. Show that if \( \ell \) and \( m \) are perpendicular then

\[
\text{slope}(\ell) = -\frac{1}{\text{slope}(m)}.
\]

c. Show that if \( \text{slope}(\ell) = -\frac{1}{\text{slope}(m)} \) then \( \ell \) and \( m \) are perpendicular.

d. What happens to the slope criterion for perpendicular lines \( \ell \) and \( m \) if one of the lines is vertical?

g. Example: Solution (DOK 2)
Suppose \( \ell \) and \( k \) are lines with slopes \( m \) and \(-\frac{1}{m}\) respectively where \( m \) is a non-zero real number. The goal of this task is to show that \( \ell \) and \( k \) are perpendicular. Below is a sample picture of \( \ell \) and \( k \) along with several marked points.

In this picture, \( \overrightarrow{PQ} \) is a horizontal line and \( \overrightarrow{OR} \) is a vertical line.

a. Suppose \( O \) is chosen as the origin and that \( P = (m, -1) \). Find the coordinates of \( Q \).

b. Show that \( \triangle QRO \) is similar to \( \triangle ORP \) with a scale factor of \( m \).

c. Deduce that \( \ell \) is perpendicular to \( k \).

h. Example: Solution (DOK 3)

a. Suppose \( \ell \) and \( m \) are parallel lines in the plane. What can you deduce about the slopes of \( \ell \) and \( m \)? Justify your answer.

b. Suppose \( \ell \) and \( m \) are distinct lines in the plane and \( \text{slope}(\ell) = \text{slope}(m) \). Are \( \ell \) and \( m \) parallel? Justify your answer.

6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio. (G-GPE.B.6) (DOK 1,2)

a. Example: Solution (DOK 2)
Below is a picture of a triangle $ABC$ on the coordinate grid. The red lines are parallel to $BC$.

Suppose $P = (1.2, 1.6)$, $Q = (2, 4)$, and $R = (2.4, 5.2)$. Find the coordinates of the points $u$, $v$, and $w$.

b. Example: Solution (DOK 3)

Below is a picture of $\triangle ABC$ with vertices lying on grid points:

- a. Draw the image of $\triangle ABC$ when it is scaled with a scale factor of $\frac{1}{3}$ about the vertex $A$: label this triangle $A'B'C''$ and find the coordinates of the points $A'$, $B'$, and $C''$.
- b. Draw the image of $\triangle ABC$ when it is scaled with a scale factor of $\frac{2}{3}$ about the vertex $B$: label this triangle $A''B''C'''$ and find the coordinates of the points $A''$, $B''$, and $C'''$.
- c. How does $A''$ compare to $B''$? Why?

7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. * (G-GPE.B.7) (DOK 1,2)

a. Example: Solution (DOK 2)
In the picture below a square is outlined whose vertices lie on the coordinate grid points:

The area of this particular square is 16 square units. For each whole number \( n \) between 1 and 10, find a square with vertices on the coordinate grid whose area is \( n \) square units or show that there is no such square.

b. Example: Solution (DOK 3)
   Below is a picture of two triangles with vertices on coordinate grid points:

   a. What are the perimeters of \( \triangle ABC \) and \( \triangle PQR \)?

   b. What is the smallest perimeter possible for a triangle with vertices on grid points and with whole number side lengths? Explain.
Explain volume formulas and use them to solve problems (G-GMD.A)

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri’s principle, and informal limit arguments. 
   **(G-GMD.A.1) (DOK 2,3)**

   a. Example: **Solution** (DOK 3)
      Suppose we define \( \pi \) to be the circumference of a circle whose diameter is 1:

      ![Diagram of a circle with diameter 1 and \( \pi \) marked]

      Explain why the circumference of a circle with radius \( r > 0 \) is \( 2\pi r \).

   b. Example: **Solution** (DOK 3)
      The goal of this task is to explain why the area enclosed by a circle \( C \) of radius \( r \) is \( \pi r^2 \). Recall that \( \pi \) is the ratio of the circumference of a circle to its diameter and that this ratio is independent of the size of the circle.

      a. Draw a picture of a regular octagon \( O \) inscribed in \( C \). Find a formula for the area of the octagon in terms of its perimeter.

      b. Reasoning as in part (a), find a formula for the area of a regular \( n \) sided polygon, for \( n \geq 3 \), inscribed in \( C \): the formula should give the area of the polygon in terms of its perimeter.

      c. Using your formula from part (b), explain why the area of \( C \) is \( \pi r^2 \).

   c. Example: **Solution** (DOK 3)
The volume of a rectangular prism of length \( l \), width \( w \), and height \( h \) is 
\[ l \times w \times h. \]

a. Use the formula for the volume of a prism with rectangular base to explain why the volume of a right prism with triangular base \( T \) and height \( h \) is \( \text{Area}(T) \times h \).

b. Use the previous part to show that the volume of a right prism with polygon base \( P \) and height \( h \) is \( \text{Area}(P) \times h \).

c. Explain why the volume of a cylinder with base \( B \) and height \( h \) is \( \text{Area}(B) \times h \).

d. Example: Solution (DOK 3)
   
The four diagonals of a cube with side length \( \ell \) meet in a point \( P \), and divide the cube into six rectangular pyramids with square bases.

   a. What is the height of each of these pyramids?

   b. What is the volume of each of these pyramids?

   c. It seems reasonable to suppose that the volume of a rectangular pyramid is, like a rectangular prism, proportional to its length \( \ell \), width \( w \), and height \( h \). That is, we expect a formula of the form
   
   \[ V = c(\ell w h) \]
   
   for some constant \( c \). Find a constant \( c \) for which this formula is true for the square pyramid described in the previous parts.

   d. Does the dissection method described in this problem work to find a formula for the volume of an arbitrary pyramid? Why?

2. (+) Give an informal argument using Cavalieri’s principle for the formulas for the volume of a sphere and other solid figures. (G-GMD.A.2) (DOK 2,3)
    a. Example: Solution (DOK 2)
The management of an ocean life museum will choose to include either Aquarium A or Aquarium B in a new exhibit.

Aquarium A is a right cylinder with a diameter of 10 feet and a height of 5 feet. Covering the lower base of Aquarium A is an “underwater mountain” in the shape of a 5-foot-tall right cone. This aquarium would be built into a pillar in the center of the exhibit room.

Aquarium B is half of a 10-foot-diameter sphere. This aquarium would protrude from the ceiling of the exhibit room.

![Aquarium A](image)

![Aquarium B](image)

a. How many cubic feet of water will Aquarium A hold?

b. For each aquarium, what is the area of the water’s surface when filled to a height of $h$ feet?

c. Use your results from parts (a) and (b) and Cavalieri’s principle to find the volume of Aquarium B.

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.* (G-GMD.A.3) [DOK 1,2]
   a. Example: Solution (DOK 2)
Janine is planning on creating a water-based centerpiece for each of the 30 tables at her wedding reception. She has already purchased a cylindrical vase for each table. The radius of the vases is 6 cm and the height is 28 cm. She intends to fill them halfway with water and then add a variety of colored marbles until the waterline is approximately three-quarters of the way up the cylinder. She can buy bags of 100 marbles in 2 different sizes, with radii of 9 mm or 12 mm. A bag of 9 mm marbles costs $3 and a bag of 12 mm marbles costs $4.

a. If Janine only bought 9 mm marbles how much would she spend on marbles for the whole reception? What if Janine only bought 12 mm marbles? (Note: 1 cm³ = 1 ml)

b. Janine’s parents, who are paying for the wedding, have told her she can spend at most $d$ dollars on marbles. Write a system of equalities and/or inequalities that she can use to determine how many marbles of each type she can buy.

c. Based on your answer to part (b), how many bags of each size marble should Janine buy if she has $180 and wants to mix in as many small marbles as possible.

b. Example: Solution (DOK 2)

Jared is scheduled for some tests at his doctor’s office tomorrow. His doctor has instructed him to drink 3 liters of water today to clear out his system before the tests. Jared forgot to bring his water bottle to work and was left in the unfortunate position of having to use the annoying paper cone cups that are provided by the water dispenser at his workplace. He measures one of these cones and finds it to have a diameter of 7 cm and a slant height (measured from the bottom vertex of the cup to any point on the opening) of 9.1 cm.

Note: 1 cm³ = 1 ml

a. How many of these cones of water must Jared drink if he typically fills the cone to within 1 cm of the top and he wants to complete his drinking during the workday?

b. Suppose that Jared drinks 25 cones of water during the day. When he gets home he measures one of his cylindrical drinking glasses and finds it to have a diameter of 7 cm and a height of 15 cm. If he typically fills his glasses to 2 cm from the top, about how many glasses of water must he drink before going to bed?

c. Example: Solution (DOK 3)
Charles and Olivia are trying to estimate the volume of water that could be held by the figure shown below, which is 10 feet high and has a circular top of radius 20 feet. Charles proposes they approximate the volume by using a cylinder of radius 20 feet and height 10 feet. Olivia proposes that they instead use a circular cone connecting the top of the tank to the vertex at the bottom.

What answers would the two methods predict? Which is likely to be most accurate? What is your best estimate for the volume of the tank?

d. Example: Solution (DOK 2)
Three of the great Egyptian pyramids are pictured below. Each is a square pyramid.

![Pyramids](image)

a. Calculate the missing information for each the 3 individual pyramids based on the given measurements:

i. The great Pyramid of Menkaure has a height of about 215 feet and a base side length of about 339 feet. What is its volume?

ii. The great Pyramid of Khafre has a volume of about 74,400,000 cubic feet and a base side length of 706 feet. What is its height?
b. The Great Pyramid of Khufu once stood 26 feet taller than it is today. Calculate the original volume of the Great Pyramid. Similarly, the Pyramid of Khafre has eroded over time and lost some of its height. If the original volume of the pyramid was approximately 78,300,000 cubic feet, what was its original height?

Visualize relationships between two-dimensional and three-dimensional objects (G-GMD.B)

1A.7. Plot points in three-dimensions. (**DOK 1**)

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects. (**G-GMD.B.4** (**DOK 1,2**)

a. Example: **Solution** (**DOK 2**)

The official diameter of a tennis ball, as defined by the International Tennis Federation, is at least 2.575 inches and at most 2.700 inches. Tennis balls are sold in cylindrical containers that contain three balls each. To model the container and the balls in it, we will assume that the balls are 2.7 inches in diameter and that the container is a cylinder the interior of which measures 2.7 inches in diameter and \( 3 \times 2.7 = 8.1 \) inches high.

a. Lying on its side, the container passes through an X-ray scanner in an airport. If the material of the container is opaque to X-rays, what outline will appear? With what dimensions?
b. If the material of the container is partially opaque to X-rays and the material of the balls is completely opaque to X-rays, what will the outline look like (still assuming the can is lying on its side)?

c. The central axis of the container is a line that passes through the centers of the top and bottom. If one cuts the container and balls by a plane passing through the central axis, what does the intersection of the plane with the container and balls look like? (The intersection is also called a cross section. Imagine putting the cut surface on an ink pad and then stamping a piece of paper. The stamped image is a picture of the intersection.)

d. If the can is cut by a plane parallel to the central axis, but at a distance of 1 inch from the axis, what will the intersection of this plane with the container and balls look like?

e. If the can is cut by a plane parallel to one end of the can—a horizontal plane—what are the possible appearances of the intersections?

f. A cross-section by a horizontal plane at a height of $1.35 + w$ inches from the bottom is made, with $0 < w < 1.35$ (so the bottom ball is cut). What is the area of the portion of the cross section inside the container but outside the tennis ball?

g. Suppose the can is cut by a plane parallel to the central axis but at a distance of $w$ inches from the axis ($0 < w < 1.35$). What fractional part of the cross section of the container is inside of a tennis ball?

b. Example: Solution (DOK 3)

Global Positioning System or GPS devices receive input from satellites and use this information to locate our position on the planet. The information received from each individual satellite gives the distance from the GPS device to that satellite and the location of the satellite. The set of points at a fixed distance from a satellite form a sphere so when the GPS receives its distance from a given satellite, this tells us that it lies on a particular sphere. Data from several satellites will locate the GPS device on the intersection of spheres. This problem examines different scenarios for intersections of spheres from the point of view of the GPS device: how many different satellites are needed to locate the GPS device? Does it matter how the satellites are configured in space?

Below is a sample picture showing three satellites and their distances, $d_1, d_2, d_3$ from a point on the earth:
The three parts of this problem investigate different positions of the three satellites pictured above: the goal is to find scenarios where one, two or three of the distances $d_1$, $d_2$, $d_3$ are (or are not) sufficient to locate the GPS unit.

a. Give an example of a satellite and GPS location so that the GPS device can determine its location on the earth with input from this single satellite. Also give an example when this one signal does not provide enough information to designate a location on the earth.

b. Give an example of a configuration of two satellites and the GPS so that the GPS device can determine its location on the earth with input from the two satellites (but not from either alone). Also give an example where these two signals do not provide enough information to designate a location on the earth.

c. Give an example of a configuration of three satellites and the GPS so that the GPS device can determine its location on the earth with input from three satellites (but not from any pair of the satellites). Also give an example where three signals do not provide enough information to designate a location on the earth.

c. Example: Solution (DOK 3)
The distance between two points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) is given by

\[\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \, .\]

a. For each of the following equations, describe the set of solutions geometrically and sketch this solution set in \(x\)-\(y\)-\(z\) coordinates:
   i. \(x^2 + y^2 + z^2 = 1\).
   ii. \((x - 3)^2 + y^2 + z^2 = 4\).
   iii. \(x^2 + y^2 + \left(z - \frac{3}{2}\right)^2 = 1\).

b. Find the set of real numbers \((x, y, z)\) which are solutions both to \(x^2 + y^2 + z^2 = 1\) and to \((x - 3)^2 + y^2 + z^2 = 4\). Sketch this solution set in \(x\)-\(y\)-\(z\) coordinates.

c. Find the set of real numbers \((x, y, z)\) which are solutions to both \(x^2 + y^2 + z^2 = 1\) and \(x^2 + y^2 + \left(z - \frac{3}{2}\right)^2 = 1\). Sketch the solution set in \(x\)-\(y\)-\(z\) coordinates.
Apply geometric concepts in modeling situations (G-MG.A)

Example: The diagram shows the end view of a roll of paper towels when it is full and the end view of the roll after some of the paper towels have been used.

When the full roll of paper towels is unrolled, it has a length of 528 inches of paper towels of uniform width and thickness. Write the length, in inches, of the paper towels remaining on the partial roll.

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1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).* (G-MG.A.1) (DOK 1,2)
   a. Example: Solution (DOK 2)
      Picture a roll of toilet paper; assume that the paper in the roll is very tightly rolled. Assuming that the paper in the roll is very thin, find a relationship between the thickness of the paper, the inner and outer radii of the roll, and the length of the paper in the roll. Express your answer as an algebraic formula involving the four listed variables.

   b. Example: Solution (DOK 3)
Suppose that even in perfect visibility conditions, the lamp at the top of a lighthouse is visible from a boat on the sea at a distance of up to 32 km (if it is farther than that, then it is obscured by the horizon).

a. If the "distance" in question is the straight-line distance from the lamp itself to the boat, what is the height above sea level of the lamp on top of the lighthouse?

b. What are two other interpretations of the distance being investigated in this problem? Describe how to solve the alternate versions.

c. Example: Solution (DOK 2)

The official diameter of a tennis ball, as defined by the International Tennis Federation, is at least 2.575 inches and at most 2.700 inches. Tennis balls are sold in cylindrical containers that contain three balls each. To model the container and the balls in it, we will assume that the balls are 2.7 inches in diameter and that the container is a cylinder the interior of which measures 2.7 inches in diameter and $3 \times 2.7 = 8.1$ inches high.

![Diagram of tennis balls in a container]

a. Lying on its side, the container passes through an X-ray scanner in an airport. If the material of the container is opaque to X-rays, what outline will appear? With what dimension?

b. If the material of the container is partially opaque to X-rays and the material of the balls is completely opaque to X-rays, what will the outline look like (still assuming the can is lying on its side)?
c. The *central axis* of the container is a line that passes through the centers of the top and bottom. If one cuts the container and balls by a plane passing through the central axis, what does the intersection of the plane with the container and balls look like? (The intersection is also called a *cross section*. Imagine putting the cut surface on an ink pad and then stamping a piece of paper. The stamped image is a picture of the intersection.)

d. If the can is cut by a plane parallel to the central axis, but at a distance of 1 inch from the axis, what will the intersection of this plane with the container and balls look like?

e. If the can is cut by a plane parallel to one end of the can—a horizontal plane—what are the possible appearances of the intersections?

f. A cross-section by a horizontal plane at a height of $1.35 + w$ inches from the bottom is made, with $0 < w < 1.35$ (so the bottom ball is cut). What is the area of the portion of the cross section inside the container but outside the tennis ball?

g. Suppose the can is cut by a plane parallel to the central axis but at a distance of $w$ inches from the axis ($0 < w < 1.35$). What fractional part of the cross section of the container is inside of a tennis ball?

d. Example: Solution (DOK 2)
The management of an ocean life museum will choose to include either Aquarium A or Aquarium B in a new exhibit.

Aquarium A is a right cylinder with a diameter of 10 feet and a height of 5 feet. Covering the lower base of Aquarium A is an "underwater mountain" in the shape of a 5-foot-tall right cone. This aquarium would be built into a pillar in the center of the exhibit room.

Aquarium B is half of a 10-foot-diameter sphere. This aquarium would protrude from the ceiling of the exhibit room.

a. How many cubic feet of water will Aquarium A hold?

b. For each aquarium, what is the area of the water's surface when filled to a height of $h$ feet?

c. Use your results from parts (a) and (b) and Cavalieri's principle to find the volume of Aquarium B.

e. Example: Solution (DOK 3)
Seven circles of the same size can be placed in the pattern shown below:

The six outer circles touch the one in the center and each circle on the outside also touches its two neighbors in the outside ring.

a. If twelve circles are placed around a central circle, as pictured below, what is the relationship between the diameters of the outer circles and the diameter of the inner circle? Explain.

b. If four circles are placed around a central circle, as pictured below, what is the relationship between the diameter of the outer circles and the diameter of the inner circle? Explain.

f. Example: Solution (DOK 3)
Beehives are made of walls, each of the same size, enclosing small hexagonal cells where honey and pollen is stored and bees are raised. Below is a picture of some cells:

![Hexagonal Cells Diagram]

The only other regular polygons which can be used to tile the plane in this way are equilateral triangles and squares. This problem examines some of the mathematical advantages of the hexagonal tiling.

Suppose we let \( s \) denote the length of the walls in the hexagonal chambers. Below is an enlarged picture of a single hexagon:

![Enlarged Hexagon Diagram]

a. Find the area of a regular hexagon \( H \) with side length \( s \).

b. Is the ratio of area to perimeter for a regular hexagon greater than or smaller than the corresponding ratios for an equilateral triangle and a square?

c. Based on your answer to (b), why do you think it is advantageous for beehives to be built using hexagonal cells instead of triangular or square cells?

g. Example: Solution (DOK 3)
Amy and Greg are raking up leaves from a large maple tree in their yard and Amy remarks "I'll bet this tree has a million leaves." Greg is skeptical. Amy suggests the following method to check whether or not this is possible:

- Find a small maple tree and estimate how many leaves it has.
- Use that number to figure out how many leaves the big maple tree has.

a. Describe the assumptions and calculations needed to carry for Amy's strategy.

b. Amy and Greg estimate that their maple tree is about 35 feet tall. They find a 5 foot tall maple tree and estimate that it has about 400 leaves. Use your work from part (a) to estimate the number of leaves on Amy and Greg's tree.

h. Example: Solution (DOK 3)

Amy and Greg look for a method to estimate the number of leaves on a 35 foot tree in their yard. Greg notices that the tree blocks almost all of the sunlight beneath its leaves so he thinks of the following way to estimate how many leaves are on the tree:

- We can first measure the area of the ground covered by the tree.
- Then we measure the area of an average leaf.
- We will need to estimate how much of its area an average leaf shades and how many leaves lie over an average point under the tree.

With all of this information we should be able to get a good estimate for the number of leaves on the tree.

a. Amy and Greg find that the tree covers a region which is roughly a circle 30 feet in diameter. What is the approximate area covered by the tree?

b. How can Amy and Greg effectively measure an irregular shape such as a tree leaf?

c. How can Amy and Greg effectively decide what number to multiply by to account for multiple leaves lying over the same area and leaves shading less than their full surface area?

d. Suppose \( a \) is the approximate area underneath the tree, \( b \) the average leaf area, \( c \) the proportion of its area an average leaf shades, and \( d \) the average number of leaves found over a spot underneath the tree. In terms of the numbers \( a, b, c, \) and \( d \) what formula will Amy and Greg find for the number of leaves on the tree?

i. Example: Solution (DOK 3)
The geometry of the earth-sun interaction plays a very prominent role in many aspects of our lives that we take for granted, like the variable length of days throughout the year and the four seasons. This problem will explore the role geometry plays in these experiences. The picture below shows the sun and the earth in four different positions of its annual orbit:
The small blue arrows show the direction of the earth’s orbit (counterclockwise) around the sun. The red line is perpendicular to the plane of the earth’s orbit and the (blue) axis of rotation is tilted approximately 23.5°. The tilt is directly toward the sun when the earth is in the position marked (A) and directly away from the sun when the earth is in the position marked (C). Here is a close up of how the sun’s rays hit the earth in position (C):

In positions (B) and (D), the tilt of the earth’s axis is neither toward the sun nor away from the sun.

In addition to orbiting around the sun once each year, the earth spins on its axis, making one complete revolution each day: it makes a little more than 365 of these revolution in each year.

a. At which point in the earth’s orbit are the days in the United States shortest? At which point in the earth’s orbit are the days in the United States the longest? Explain.

b. Indicate in the picture which sections of the orbit correspond to the four seasons (winter, spring, summer, fall) in the United States. Justify your choices.

c. The tropics are the region of the earth where the sun’s rays meet the earth perpendicularly at some point in the year. Explain why the tropics are the area between the lines of latitude of about 23.5° in the northern and southern hemispheres.

j. Example: Solution (DOK 3)
A solar eclipse occurs when the moon passes between the sun and the earth. The eclipse is called partial if the moon only blocks part of the sun and total if the moon blocks out the entire sun. A picture (not drawn to scale) of the configuration of the earth, moon, and sun during a total eclipse at the point of the earth marked in red is shown below with the earth in blue, the moon black, and the sun yellow:

![Schematic diagram of solar eclipse](image)

The sun is approximately 93,000,000 miles from the earth and the moon is about 239,000 miles from the earth, with a range from about 225,600 miles at its closest to 252,100 at its furthest. The radius of the moon is approximately 1080 miles.

a. Total eclipses are rare but do occur: given this information, how large could the radius of the sun be?

b. The radius of the sun is approximately 432,000 miles. What is the furthest the moon can be from the earth for a total eclipse to occur? Does this help to explain why total eclipses are rare?

k. Example: Solution (DOK 2)

About how many cells are in the human body?

You can assume that a cell is a sphere with radius $10^{-3}$ cm and that the density of a cell is approximately the density of water which is 1g/cm$^3$.

l. Example: Solution (DOK 2)
A cylindrical soda can is made of aluminum. It is approximately 4 \( \frac{3}{4} \) inches high and the top and bottom have a radius of approximately 1 \( \frac{3}{10} \) inches:

a. Find the approximate surface area of the soda can. What assumptions do you use in your estimate?

b. The density of aluminum is approximately 2.70 grams per cubic centimeter. If the mass of the soda can is approximately 15 grams, how many cubic centimeters of aluminum does it contain?

c. Using the answers to (a) and (b) estimate how thick the aluminum can is.

m. Example: Solution (DOK 3)

About how thick is a soda can?

Explain which measurements you will need to take as well as extra information that you may need in order to estimate the thickness of a soda can. You may assume that the can is made of aluminum. Because of the risk of injury, cutting the can and directly measuring its thickness is not permitted.

n. Example: Solution (DOK 3)
Global Positioning System or GPS devices receive input from satellites and use this information to locate our position on the planet. The information received from each individual satellite gives the distance from the GPS device to that satellite and the location of the satellite. The set of points at a fixed distance from a satellite form a sphere so when the GPS receives its distance from a given satellite, this tells us that it lies on a particular sphere. Data from several satellites will locate the GPS device on the intersection of spheres. This problem examines different scenarios for intersections of spheres from the point of view of the GPS device: how many different satellites are needed to locate the GPS device? Does it matter how the satellites are configured in space?

Below is a sample picture showing three satellites and their distances, $d_1, d_2, d_3$ from a point on the earth:

The three parts of this problem investigate different positions of the three satellites pictured above: the goal is to find scenarios where one, two or three of the distances $d_1, d_2, d_3$ are (or are not) sufficient to locate the GPS unit.

a. Give an example of a satellite and GPS location so that the GPS device can determine its location on the earth with input from this single satellite. Also give an example when this one signal does not provide enough information to designate a location on the earth.
b. Give an example of a configuration of two satellites and the GPS so that the GPS device can determine its location on the earth with input from the two satellites (but not from either alone). Also give an example where these two signals do not provide enough information to designate a location on the earth.

c. Give an example of a configuration of three satellites and the GPS so that the GPS device can determine its location on the earth with input from three satellites (but not from any pair of the satellites). Also give an example where three signals do not provide enough information to designate a location on the earth.

o. Example; Solution (DOK 3)

   Milong and her friends are at the beach looking out onto the ocean on a clear day and they wonder how far away the horizon is.

   a. About how far can Milong see out on the ocean?

   b. If Milong climbs up onto a lifeguard tower, how far is the horizon in Milong’s view?

   c. Mount Shishaldin lies on a narrow peninsula in Alaska and is pictured here:

   ![Mount Shishaldin](image)

   If Milong were to stand atop Mount Shishaldin and look out over the ocean, how far would the horizon be?

   d. Based on the answers to the questions above, is the distance of Milong’s visual horizon proportional to her elevation above the surface of the earth? Explain.

2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). * (G-MG.A.2) (DOK 1,2)

3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). * (G-MG.A.3) (DOK 2,3,4)
(IA) Use diagrams consisting of vertices and edges (vertex-edge graphs) to model and solve problems related to networks.

IA.8. Understand, analyze, evaluate, and apply vertex-edge graphs to model and solve problems related to paths, circuits, networks, and relationships among a finite number of elements, in real-world and abstract settings. *(DOK 2,3)*

IA.9. Model and solve problems using at least two of the following fundamental graph topics and models: Euler paths and circuits, Hamilton paths and circuits, the traveling salesman problem (TSP), minimum spanning trees, critical paths, vertex coloring. *(DOK 2,3)*

IA.10. Compare and contrast vertex-edge graph topics and models in terms of: *(DOK 2,3)*

- properties
- algorithms
- optimization
- types of problems that can be solved
Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data; in critically reviewing uses of statistics in public media and other reports, it is important to consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn.

Random processes can be described mathematically by using a probability model: a list or description of the possible outcomes (the sample space), each of which is assigned a probability. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the Addition and Multiplication Rules. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables.
Technology plays an important role in statistics and probability by making it possible to generate plots, regression functions, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

**Connections to Functions and Modeling.** Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be expressed through a correlation coefficient.
Statistics and Probability Overview

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable
- Summarize, represent, and interpret data on two categorical and quantitative variables
- Interpret linear models

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments
- Make inferences and justify conclusions from sample surveys, experiments and observational studies

Conditional Probability and the Rules of Probability

- Understand independence and conditional probability and use them to interpret data
- Use the rules of probability to compute probabilities of compound events in a uniform probability model

Using Probability to Make Decisions

- Calculate expected values and use them to solve problems
- Use probability to evaluate outcomes of decisions

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
Interpreting Categorical and Quantitative Data

Summarize, represent, and interpret data on a single count or measurement variable (S-ID.A)

Example: The dot plots below compare the number of minutes 30 flights made by two airlines arrived before or after their scheduled arrival times.

- Negative numbers represent the minutes the flight arrived before its scheduled time.
- Positive numbers represent the minutes the flight arrived after its scheduled time.
- Zero indicates the flight arrived at its scheduled time.

Assuming you want to arrive as close to the scheduled time as possible, from which airline should you buy your ticket? Use the ideas of center and spread to justify your choice.

<table>
<thead>
<tr>
<th>Item</th>
<th>Claim</th>
<th>Domain</th>
<th>Target</th>
<th>DOK</th>
<th>CONTENT</th>
<th>MP</th>
<th>Key</th>
</tr>
</thead>
<tbody>
<tr>
<td>#18</td>
<td>4</td>
<td>S-ID</td>
<td>B</td>
<td>3</td>
<td>HSS.ID.A</td>
<td>2, 3</td>
<td>See exemplar</td>
</tr>
</tbody>
</table>

Exemplar:
I would buy the ticket from Airline P. Both airlines are likely to have an on-time arrival since they both have median values at 0. However, Airline Q has a much greater range in arrival times. Airline Q could arrive anywhere from 35 minutes early to 60 minutes late. For Airline P, the flights arrived within 10 minutes on either side of the scheduled arrival time about 2/3 of the time, and for Airline Q, that number was only about 1/2. For these reasons, I think Airline P is the better choice.

Rubric:
(2 points) Student chooses Airline P and clearly explains that both airlines have the same center but that Airline P has a smaller spread.

(1 point) Student states that either airline could be chosen because they have the same median, but does not address the issue of spread; OR The student states that both airlines have the same median and chooses Airline P, but does not justify the choice based on spread; OR The student explains that Airline P would be the better choice based on the smaller spread, but does not identify that both airlines have the same median.
1. Represent data with plots on the real number line (dot plots, histograms, and box plots). *(S-ID.A.1) (DOK 1,2)*

   a. Example: Solution (DOK 3)

   Seventy-five female college students and 24 male college students reported the cost (in dollars) of his or her most recent haircut. The resulting data are summarized in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Females</th>
<th>Males</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of observations</td>
<td>75</td>
<td>24</td>
</tr>
<tr>
<td>Minimum</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Maximum</td>
<td>150</td>
<td>35</td>
</tr>
<tr>
<td>1st Quartile</td>
<td>20</td>
<td>9.25</td>
</tr>
<tr>
<td>Median</td>
<td>31</td>
<td>17</td>
</tr>
<tr>
<td>3rd Quartile</td>
<td>75</td>
<td>20</td>
</tr>
<tr>
<td>Mean</td>
<td>52.53</td>
<td>20.13</td>
</tr>
</tbody>
</table>

   a. Using the minimum, maximum, quartiles and median, sketch two side by side box plots to compare the hair cut costs between males and females in this student's school.

   b. How would you describe the difference in haircut costs between males and females? Be sure you discuss differences/similarities in shape, center and spread.

   c. Why is the mean greater than the median both for males and for females? Explain your reasoning.

   d. Is the median or mean a more appropriate choice for describing the “centers” of these two distributions?
b. Example: Solution (DOK 2)

A statistically-minded state trooper wondered if the speed distributions are similar for cars traveling northbound and for cars traveling southbound on an isolated stretch of interstate highway. He uses a radar gun to measure the speed of all northbound cars and all southbound cars passing a particular location during a fifteen minute period. Here are his results:

<table>
<thead>
<tr>
<th>Northbound Cars</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
</tr>
<tr>
<td>63</td>
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<tr>
<td>65</td>
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<tr>
<td>67</td>
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<td>62</td>
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<tr>
<td>64</td>
</tr>
<tr>
<td>66</td>
</tr>
<tr>
<td>83</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Southbound Cars</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
</tr>
<tr>
<td>60</td>
</tr>
<tr>
<td>64</td>
</tr>
<tr>
<td>68</td>
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<tr>
<td>56</td>
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<td>57</td>
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<tr>
<td>62</td>
</tr>
<tr>
<td>67</td>
</tr>
<tr>
<td>71</td>
</tr>
</tbody>
</table>

Draw box plots of these two data sets, and then use the plots and appropriate numerical summaries of the data to write a few sentences comparing the speeds of northbound cars and southbound cars at this location during the fifteen minute time period.

c. Example: Shade in the circles to create a dot plot for the given test scores. 90, 45, 85, 70, 85, 50, 75, 85, 65, 75, 60, 85, 80, 65, 80
d. Example: SPEEDING TICKETS

New York State wants to change its system for assigning speeding fines to drivers. The current system allows a judge to assign a fine that is within the ranges shown in Table 1.

**Table 1. New York Speeding Fines**

<table>
<thead>
<tr>
<th>Miles per Hour over Speed Limit</th>
<th>Minimum Fine</th>
<th>Maximum Fine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 – 10</td>
<td>$45</td>
<td>$150</td>
</tr>
<tr>
<td>11 – 30</td>
<td>$90</td>
<td>$300</td>
</tr>
<tr>
<td>31 or more</td>
<td>$180</td>
<td>$600</td>
</tr>
</tbody>
</table>

Some people have complained that the New York speeding fine system is not fair. The New Drivers Association (NDA) is recommending a new speeding fine system. The NDA is studying the Massachusetts system because of claims that it is fairer than the New York system.

**Table 2. Massachusetts Speeding Fines**

<table>
<thead>
<tr>
<th>Miles per Hour over Speed Limit</th>
<th>Fine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 – 10</td>
<td>$100 flat charge</td>
</tr>
<tr>
<td>11 or more</td>
<td>$100 flat charge plus $10 for each additional mph above the first 10 mph</td>
</tr>
</tbody>
</table>

In this task, you will:

- Analyze the speeding fine systems for both New York and Massachusetts.
- Use data to propose a fairer speeding fine system for New York State.
Part A
Use the information in Table 2 to plot data points for Massachusetts speeding fines.
- Plot a point to represent the fine for driving 5 mph over the speed limit.
- Plot additional points for each increment of 5 mph over the speed limit up to 45 mph over the speed limit.

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Table 2. Massachusetts Speeding Fines
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. (S-ID.A.2) (DOK 1,2)
   a. Example: Solution (DOK 3)

   Seventy-five female college students and 24 male college students reported the cost (in dollars) of his or her most recent haircut. The resulting data are summarized in the following table.

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   a. Using the minimum, maximum, quartiles and median, sketch two side by side box plots to compare the hair cut costs between males and females in this student's school.

   b. How would you describe the difference in haircut costs between males and females? Be sure you discuss differences/similarities in shape, center and spread.

   c. Why is the mean greater than the median both for males and for females? Explain your reasoning.

   d. Is the median or mean a more appropriate choice for describing the “centers” of these two distributions?
b. Example: Solution (DOK 2)

A statistically-minded state trooper wondered if the speed distributions are similar for cars traveling northbound and for cars traveling southbound on an isolated stretch of interstate highway. He uses a radar gun to measure the speed of all northbound cars and all southbound cars passing a particular location during a fifteen minute period. Here are his results:

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<td></td>
<td>67</td>
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<td>79</td>
<td>83</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Southbound Cars</th>
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</tr>
</thead>
<tbody>
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<td></td>
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<td>68</td>
<td>68</td>
<td>68</td>
<td>71</td>
</tr>
</tbody>
</table>

Draw box plots of these two data sets, and then use the plots and appropriate numerical summaries of the data to write a few sentences comparing the speeds of northbound cars and southbound cars at this location during the fifteen minute time period.

c. Example: Solution (DOK 3)
This task is divided into four parts.

**Part 1**

Below are dot plots for three different data sets. The standard deviations for these three data sets are given in the following table. Looking at the dot plots and without calculating the standard deviations, match the data sets to the standard deviations.

<table>
<thead>
<tr>
<th>Standard deviation</th>
<th>Data Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.9</td>
<td></td>
</tr>
<tr>
<td>3.3</td>
<td></td>
</tr>
<tr>
<td>2.3</td>
<td></td>
</tr>
</tbody>
</table>

**Part 2**

a. Draw 4 rectangles for which the standard deviation of the 4 rectangle heights would be equal to 0.

b. Draw 4 rectangles for which the standard deviation of the 4 rectangle heights would be greater than the standard deviation of the rectangle widths.
Part 3

Which of the two histograms below represents the data distribution with the greater standard deviation? Explain your choice.

Part 4

a. Write two sets of 5 different numbers that have the same mean but different standard deviations.

b. Write two sets of 5 different numbers that have the same standard deviations but different means.

d. Example: Solution (DOK 3)
Jim has taken 4 exams in his Statistics class.

<table>
<thead>
<tr>
<th>Exam</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>92</td>
</tr>
<tr>
<td>2</td>
<td>84</td>
</tr>
<tr>
<td>3</td>
<td>96</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
</tr>
</tbody>
</table>

a. Both the MAD (mean absolute deviation) and the standard deviation are commonly used to measure variability in a data set. Explain how the way the standard deviation is calculated is different than the way the MAD is calculated.

b. Calculate the standard deviation of Jim’s scores and explain how this value represents the variability in his test scores.

c. Sally took the same exams and also had a mean score of 93. Both students’ scores are displayed in the dot plots below. Without calculating, determine if Sally’s standard deviation is larger than, smaller than or equal to the standard deviation for Jim’s scores. Explain your reasoning.

d. Tom also took the same exams. His mean was 91 and his standard deviation was zero. What scores did Tom receive on his exams?

3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). (S-ID.A.3) (DOK 1,2)
   a. Example: Solution (DOK 3)
Seventy-five female college students and 24 male college students reported the cost (in dollars) of his or her most recent haircut. The resulting data are summarized in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Females</th>
<th>Males</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of observations</td>
<td>75</td>
<td>24</td>
</tr>
<tr>
<td>Minimum</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Maximum</td>
<td>150</td>
<td>35</td>
</tr>
<tr>
<td>1st Quartile</td>
<td>20</td>
<td>9.25</td>
</tr>
<tr>
<td>Median</td>
<td>31</td>
<td>17</td>
</tr>
<tr>
<td>3rd Quartile</td>
<td>75</td>
<td>20</td>
</tr>
<tr>
<td>Mean</td>
<td>52.53</td>
<td>20.13</td>
</tr>
</tbody>
</table>

a. Using the minimum, maximum, quartiles and median, sketch two side by side box plots to compare the hair cut costs between males and females in this student's school.

b. How would you describe the difference in hair cut costs between males and females? Be sure you discuss differences/similarities in shape, center and spread.

c. Why is the mean greater than the median both for males and for females? Explain your reasoning.

d. Is the median or mean a more appropriate choice for describing the "centers" of these two distributions?

b. Example: Solution (DOK 2)
A statistically-minded state trooper wondered if the speed distributions are similar for cars traveling northbound and for cars traveling southbound on an isolated stretch of interstate highway. He uses a radar gun to measure the speed of all northbound cars and all southbound cars passing a particular location during a fifteen minute period. Here are his results:

<table>
<thead>
<tr>
<th>Northbound Cars</th>
</tr>
</thead>
<tbody>
<tr>
<td>60 62 62 63 63</td>
</tr>
<tr>
<td>63 64 64 64 65</td>
</tr>
<tr>
<td>65 65 66 66 66</td>
</tr>
<tr>
<td>67 68 70 83 83</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Southbound Cars</th>
</tr>
</thead>
<tbody>
<tr>
<td>55 56 57 57 58</td>
</tr>
<tr>
<td>60 61 61 62 63</td>
</tr>
<tr>
<td>64 65 65 67 67</td>
</tr>
<tr>
<td>68 68 68 68 71</td>
</tr>
</tbody>
</table>

Draw box plots of these two data sets, and then use the plots and appropriate numerical summaries of the data to write a few sentences comparing the speeds of northbound cars and southbound cars at this location during the fifteen minute time period.

c. Example: **Solution** (DOK 3)

For certain data sets, such as home prices and household or individual income, is often described using the median instead of the mean. The questions below explore the mean and median in some different situations to help you understand the information that they communicate.

a. Give an example of a set of five positive numbers whose median is 10 and whose mean is larger than 10.

b. Find the mean and median of the following set of numbers: {10, 15, 25, 30, 30, 50, 55, 55, 60, 80}. What happens to the mean and median of these numbers if 80 is replaced by 800?
c. The brightness of celestial bodies depends on many factors, two of the most important being the distance from Earth and size. The eight brightest objects in the night sky are listed below with their approximate distance from Earth (in light years).

<table>
<thead>
<tr>
<th>Object</th>
<th>Distance in light years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moon</td>
<td>0.000000038</td>
</tr>
<tr>
<td>Venus</td>
<td>0.0000048</td>
</tr>
<tr>
<td>Jupiter</td>
<td>0.0000027</td>
</tr>
<tr>
<td>Mars</td>
<td>0.0000076</td>
</tr>
<tr>
<td>Mercury</td>
<td>0.0000095</td>
</tr>
<tr>
<td>Sirius</td>
<td>8.6</td>
</tr>
<tr>
<td>Canopus</td>
<td>310</td>
</tr>
<tr>
<td>Saturn</td>
<td>0.00014</td>
</tr>
</tbody>
</table>

Calculate the mean and median for these distances. Would the typical distance of these celestial bodies best be communicated using the mean or the median? Why?

d. What impact do the very large values in the data set have on the mean?

e. Suppose that a sample of 100 homes in the metropolitan Phoenix area had a median sales price of $300,000. The mean value of these homes was $1,000,000. Explain how this could happen. Why might the median price be more informative than the mean price in describing a typical house price?

f. Suppose the mean annual income for a sample of one hundred Minneapolis residents was $50,000. Do you think the median income for this sample would have been greater than, equal to, or less than $50,000? Explain.

d. Example: Solution (DOK 3)
Jim has taken 4 exams in his Statistics class.

<table>
<thead>
<tr>
<th>Exam</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>92</td>
</tr>
<tr>
<td>2</td>
<td>84</td>
</tr>
<tr>
<td>3</td>
<td>96</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
</tr>
</tbody>
</table>

a. Both the MAD (mean absolute deviation) and the standard deviation are commonly used to measure variability in a data set. Explain how the way the standard deviation is calculated is different than the way the MAD is calculated.

b. Calculate the standard deviation of Jim's scores and explain how this value represents the variability in his test scores.

c. Sally took the same exams and also had a mean score of 93. Both students' scores are displayed in the dot plots below. Without calculating, determine if Sally's standard deviation is larger than, smaller than or equal to the standard deviation for Jim's scores. Explain your reasoning.

d. Tom also took the same exams. His mean was 91 and his standard deviation was zero. What scores did Tom receive on his exams?

e. Example: Solution (DOK 3)
Students were asked to report how far (in miles) they each live from school. The following distances were recorded.

<table>
<thead>
<tr>
<th>Student</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Zane</td>
</tr>
<tr>
<td>2</td>
<td>Jackson</td>
</tr>
<tr>
<td>3</td>
<td>Benjamin</td>
</tr>
<tr>
<td>4</td>
<td>Bethany</td>
</tr>
<tr>
<td>5</td>
<td>Joe</td>
</tr>
<tr>
<td>6</td>
<td>Noelle</td>
</tr>
<tr>
<td>7</td>
<td>Tianye</td>
</tr>
<tr>
<td>8</td>
<td>Anthony</td>
</tr>
<tr>
<td>9</td>
<td>Amanda</td>
</tr>
<tr>
<td>10</td>
<td>Michaela</td>
</tr>
<tr>
<td>11</td>
<td>Miranda</td>
</tr>
<tr>
<td>12</td>
<td>Joseph</td>
</tr>
<tr>
<td>13</td>
<td>John</td>
</tr>
</tbody>
</table>

1. Summary statistics for the distances are given below.

<table>
<thead>
<tr>
<th>Min</th>
<th>Q1</th>
<th>Median</th>
<th>Q3</th>
<th>Max</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>1.0</td>
<td>3.0</td>
<td>4.3</td>
<td>7.5</td>
<td>3.03</td>
</tr>
</tbody>
</table>
Construct a box plot for the distances and describe the main features of the distribution.

2. John currently lives furthest from school. Would you consider his distance from school (7.5 miles) to be “unusual”? Explain.

3. John’s family is considering moving to a house that is 10 miles from school. How will this move affect the summary statistics? Change John’s distance from 7.5 miles to 10 miles and complete the table based on this new data.

<table>
<thead>
<tr>
<th>Min</th>
<th>Q1</th>
<th>Median</th>
<th>Q3</th>
<th>Max</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Has the mean increased, decreased or remained unchanged? Has the median increased, decreased or remained unchanged? Explain how John’s move has affected these measures of center.

4. Construct a boxplot of the distances after John’s move. Would you consider John’s new distance from school (10 miles) to be “unusual” now? Explain.

5. A data point can be considered an “outlier” if it is more than 1.5 times the IQR above Q3 or more than 1.5 times the IQR below Q1. Using this description of an outlier, was John’s distance from school considered an outlier before the move? How about after the move?

6. Zane lives closest to school. Using the description of an outlier given in question 5, is his distance considered an outlier?

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. (S-ID.A.4) (DOK 1,2)
   a. Example: Solution (DOK 2)
Suppose that SAT mathematics scores for a particular year are approximately normally distributed with a mean of 510 and a standard deviation of 100.

a. What is the probability that a randomly selected score is greater than 610?

b. Greater than 710?

c. Between 410 and 710?

d. If a student is known to score 750, what is the student’s percentile score (the proportion of scores below 750)?

b. Example: Solution (DOK 2)

   Automobile manufacturers have to design the driver’s seat area so that both tall and short adults can sit comfortably, reach all the controls and pedals, and see through the windshield. Suppose a new car is designed so that these conditions are met for people from 58 inches to 76 inches tall.

   The heights of adult men in the United States are approximately normally distributed with a mean of 70 inches and a standard deviation of 3 inches. Heights of adult women are approximately normally distributed with a mean of 64.5 inches and a standard deviation of 2.5 inches. What percentage of men in the United States is this car not designed to accommodate? What percentage of women in the United States is this car not designed to accommodate?

c. Example: Solution (DOK 3)
Test scores on a statewide standardized test for a large population of students are normally distributed with mean = 9.44 and standard deviation = 1.75.

a. Approximately what percentage of the scores are between 7.69 and 11.19?

b. Certificates are given to students who score in the top 2.5% of those who took the test. Fred, a student who took the test, finds out that he earned a score of 13.1 on the test. He wonders if he should have received a certificate in the mail by now. He contacts the company that administers the test and asks if his score was high enough to earn a certificate.

Imagine that you work for this company that administers the test, and your supervisor (Chris) asks you to look into the matter. Complete the following note to Chris that clearly states if Fred is to receive a certificate and includes a brief summary of your analysis that led you to that conclusion. Assume that your supervisor, Chris, is familiar with z-scores, probabilities, normal curves, etc.

Chris:
Regarding your request about Fred’s test score, ...

Summarize, represent, and interpret data on two categorical and quantitative variables (S-ID.B)

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. (S-ID.B.5) (DOK 1,2)

   a. Example: Solution (DOK 3)
The 54 students in one of several middle school classrooms were asked two questions about musical preferences: “Do you like rock?” “Do you like rap?” The responses are summarized in the table below.

<table>
<thead>
<tr>
<th></th>
<th>Likes Rap</th>
<th>Doesn’t Like Rap</th>
<th>Row Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Likes Rock</td>
<td>27</td>
<td>6</td>
<td>33</td>
</tr>
<tr>
<td>Doesn’t Like Rock</td>
<td>4</td>
<td>17</td>
<td>21</td>
</tr>
<tr>
<td>Column Totals</td>
<td>31</td>
<td>23</td>
<td>54</td>
</tr>
</tbody>
</table>

a. Is this a random sample, one that fairly represents the opinions of all students in the middle school?

b. What percentage of the students in the classroom like rock?

c. Is there evidence in this sample of a positive association between liking rock and liking rap? Justify your answer by pointing out a feature of the table that supports it.

d. Explain why the results for this classroom might not generalize to the entire middle school.

b. Example: **Solution** (DOK 3)

Each student in a random sample of students at a local high school was categorized according to gender (male or female) and whether they supported a proposal to increase the length of the school day by 30 minutes (oppose, support, no opinion). The following table summarizes the data for this sample.

<table>
<thead>
<tr>
<th>Opinion on Proposal to Increase Length of School Day</th>
<th>Oppose</th>
<th>Support</th>
<th>No Opinion</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>50</td>
<td>40</td>
<td>20</td>
<td>110</td>
</tr>
<tr>
<td>Female</td>
<td>40</td>
<td>40</td>
<td>10</td>
<td>90</td>
</tr>
<tr>
<td>Total</td>
<td>90</td>
<td>80</td>
<td>30</td>
<td>200</td>
</tr>
</tbody>
</table>
a. What proportion of the students in this sample are male?

b. What proportion of the students in this sample support the proposal?

c. What proportion of the males in this sample support the proposal?

d. What proportion of the students in this sample who support this proposal are male?

e. Interpret the following joint relative frequency in the context of this problem: 10/200.

f. Interpret the following marginal relative frequency in the context of this problem: 30/200.

g. Interpret the following conditional frequency in the context of this problem: 50/110.

h. Interpret the following conditional frequency in the context of this problem: 20/110.

i. Interpret the following conditional frequency in the context of this problem: 20/30.

6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.
   b. Informally assess the fit of a function by plotting and analyzing residuals.
   c. Fit a linear function for a scatter plot that suggests a linear association. (S-ID.B.6) (DOK 1,2)
      1. Example: Solution (DOK 3)
<table>
<thead>
<tr>
<th>Age</th>
<th>Mileage</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>109428</td>
<td>12995</td>
</tr>
<tr>
<td>5</td>
<td>84804</td>
<td>14588</td>
</tr>
<tr>
<td>3</td>
<td>55321</td>
<td>20994</td>
</tr>
<tr>
<td>3</td>
<td>57474</td>
<td>18991</td>
</tr>
<tr>
<td>1</td>
<td>11696</td>
<td>19981</td>
</tr>
<tr>
<td>13</td>
<td>125260</td>
<td>6888</td>
</tr>
<tr>
<td>10</td>
<td>67740</td>
<td>9888</td>
</tr>
<tr>
<td>11</td>
<td>97500</td>
<td>6950</td>
</tr>
<tr>
<td>6</td>
<td>36967</td>
<td>19700</td>
</tr>
<tr>
<td>12</td>
<td>148000</td>
<td>3995</td>
</tr>
<tr>
<td>2</td>
<td>29836</td>
<td>18990</td>
</tr>
<tr>
<td>3</td>
<td>32349</td>
<td>21995</td>
</tr>
<tr>
<td>10</td>
<td>161460</td>
<td>5995</td>
</tr>
<tr>
<td>4</td>
<td>68075</td>
<td>12999</td>
</tr>
<tr>
<td>3</td>
<td>30007</td>
<td>22900</td>
</tr>
<tr>
<td>8</td>
<td>66000</td>
<td>13995</td>
</tr>
<tr>
<td>10</td>
<td>93450</td>
<td>8488</td>
</tr>
<tr>
<td>3</td>
<td>35518</td>
<td>22995</td>
</tr>
<tr>
<td>3</td>
<td>30047</td>
<td>20850</td>
</tr>
<tr>
<td>8</td>
<td>107506</td>
<td>11988</td>
</tr>
</tbody>
</table>
2. Example: Solution (DOK 3)

Many counties in the United States are governed by a county council. At public county council meetings, county residents are usually allowed to bring up issues of concern. At a recent public County Council meeting, one resident expressed concern that 3 new coffee shops from a popular coffee shop chain were planning to open in the county, and the resident believed that this would create an increase in property crimes in the county. (Property crimes include burglary, larceny-theft, motor vehicle theft, and arson -- From http://www.fbi.gov/about-us/cjis/ucr/crime-in-the-u.s/2010/crime-in-the-u.s.-2010/property-crime accessed on December 5, 2012.)

To support this claim, the resident presented the following data and scatterplot (with the least-squares line shown) for 8 counties in the state:

<table>
<thead>
<tr>
<th>County</th>
<th>Shops</th>
<th>Crimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>9</td>
<td>4000</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>2700</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>500</td>
</tr>
<tr>
<td>D</td>
<td>6</td>
<td>4200</td>
</tr>
<tr>
<td>E</td>
<td>15</td>
<td>6800</td>
</tr>
<tr>
<td>F</td>
<td>50</td>
<td>20800</td>
</tr>
<tr>
<td>G</td>
<td>5</td>
<td>2800</td>
</tr>
<tr>
<td>H</td>
<td>24</td>
<td>15400</td>
</tr>
</tbody>
</table>
The scatterplot shows a positive linear relationship between "Shops" (the number of coffee shops of this coffee shop chain in the county) and "Crimes" (the number of annual property crimes for the county). In other words, counties with more of these coffee shops tend to have more property crimes annually.

a. Does the relationship between Shops and Crimes appear to be linear? Would you consider the relationship between Shops and Crimes to be strong, moderate, or weak?

b. Compute the correlation coefficient. Does the value of the correlation coefficient support your choice in part (a)? Explain.

c. The equation of the least-squares line for these data is:

\[
\text{Predicted Crimes} = 1434 + 415.7(\text{Shops})
\]

Based on this line, what is the estimated number of additional annual property crimes for a given county that has 3 more coffee shops than another county?

d. Do these data support the claim that building 3 additional coffee shops will necessarily cause an increase in property crimes? What other variables might explain the positive relationship between the number of coffee shops for this coffee shop chain and the number of annual property crimes for these counties?

e. If the following two counties were added to the data set, would you still consider using a line to model the relationship? If not, what other types (forms) of model would you consider?

<table>
<thead>
<tr>
<th>County</th>
<th>Shops</th>
<th>Crimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>25</td>
<td>36900</td>
</tr>
<tr>
<td>J</td>
<td>27</td>
<td>24100</td>
</tr>
</tbody>
</table>

3. Example: Solution (DOK 3)
The scatterplot below shows the finishing times for the Olympic gold medalist in the men's 100-meter dash for many previous Olympic games. The line of best fit is also shown. (Source: http://trackandfield.about.com/od/sprintsandrelays/qt/olympic100medals.htm)

![Olympic Gold Medalist — Men's 100-m Dash](image)

a. Is a linear model a good fit for the data? Explain, commenting on the strength and direction of the association.

b. The equation of the linear function that best fits the data (the line of best fit) is

\[
\text{Finishing time} = 10.878 - 0.0106 (\text{Year after 1900})
\]

Given that the summer Olympic games only take place every four years, how should we expect the gold medalist's finishing time to change from one Olympic games to the next?

c. What is the vertical intercept of the function's graph? What does it mean in context of the 100-meter dash?

d. Note that the gold medalist finishing time for the 1940 Olympic games is not included in the scatterplot. Use the model to estimate the gold medalist's finishing time for that year.

e. What is a realistic domain for the linear function? Comment on how your answer pertains to using this function to make predictions about future Olympic 100-m dash race times.

4. Example: Solution (DOK 3)
Jerry forgot to plug in his laptop before he went to bed. He wants to take the laptop to his friend’s house with a full battery. The pictures below show screenshots of the battery charge indicator after he plugs in the computer.

a. When can Jerry expect that his laptop battery is fully charged?

b. At 9:27 AM Jerry makes a quick calculation:

   The battery seems to be charging at a rate of 1 percentage point per minute. So the battery should be fully charged at 10:11 AM.

Explain Jerry’s calculation. Is his estimate most likely an under- or over-estimate? How does it compare to your prediction?

c. Compare the average rate of change of the battery charging function on the first given time interval and on the last given time interval. What does this tell you about how the battery is charging?

d. How long would it take for the battery to charge if it started out completely empty?

5. Example: Solution (DOK 3)
The owner of a local restaurant selected a random sample of dinner tables at his restaurant. For each table, the owner recorded the total amount of the dinner bill and the number of people at the table. The data are given in the table below. A scatterplot of the data is also shown.

<table>
<thead>
<tr>
<th>People</th>
<th>Bill</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$8.50</td>
</tr>
<tr>
<td>2</td>
<td>$36.00</td>
</tr>
<tr>
<td>2</td>
<td>$32.55</td>
</tr>
<tr>
<td>3</td>
<td>$29.30</td>
</tr>
<tr>
<td>3</td>
<td>$50.65</td>
</tr>
<tr>
<td>4</td>
<td>$48.75</td>
</tr>
<tr>
<td>5</td>
<td>$63.75</td>
</tr>
<tr>
<td>5</td>
<td>$60.00</td>
</tr>
<tr>
<td>8</td>
<td>$82.50</td>
</tr>
<tr>
<td>10</td>
<td>$125.75</td>
</tr>
</tbody>
</table>
a. Does the relationship between number of people and the bill appear to be weak, moderate or strong?

b. Does the relationship between number of people and the bill appear to be linear?

c. The equation of the line of best fit is $\text{Bill} = 5.80 + 11.15 \cdot \text{People}$. Sketch this line on the scatterplot.

d. Interpret the slope of the line of best fit in the context of this problem.

e. Note that the points in the scatterplot do not all lie on the line. What could explain this variability?

f. Use the equation of the line of best fit to estimate how much you predict the bill to be for a party of 8?

g. There was one party of 8 whose bill was $82.50. Did they pay more than or less than the predicted amount? How much more or how much less did they pay?
Interpret linear models (S-ID.C)

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. (S-ID.C.7) (DOK 1,2)
   a. Example: Solution (DOK 3)
Jane wants to sell her Subaru Forester and does research online to find other cars for sale in her area. She checks on craigslist.com and finds 22 Subaru Foresters recently listed, along with their mileage (in miles), age (in years), and listed price (in dollars). (Collected on June 6th, 2012 for the San Francisco Bay Area.)

She examines the scatterplot of price versus age and determines that a linear model is appropriate. She finds the equation of the least squares regression equation:

\[ \text{predicted price} = 24,247.56 - 1482.06 \text{ age}. \]

a. What variable is the explanatory (independent) variable and what are the units it is measured in? What variable is the response (dependent) variable and what are the units it is measured in?

b. What is the slope of the least squares regression line and what are its units?

c. Interpret the slope of the least squares regression line in the context of the problem, discussing what the slope tells you about how price and age are related. Use appropriate units in your answer.

d. What is the \( y \)-intercept of the least squares regression line? Interpret the \( y \)-intercept in the context of the problem, including appropriate units.

b. Example: Solution (DOK 3)
Medhavi suspects that there is a relationship between the number of text messages high school students send and their academic achievement. To explore this, she asks a random sample of 52 students at her school how many text messages they sent yesterday and what their grade point average (GPA) was during the most recent marking period. Her data are summarized in the scatter plot below. The line of best fit is also shown.

The equation of the line of best fit is $GPA = 3.8 - 0.005(\text{Texts sent})$. Interpret the quantities $-0.005$ and $3.8$ in the context of these data.

c. Example: Solution (DOK 3)
Many counties in the United States are governed by a county council. At public county council meetings, county residents are usually allowed to bring up issues of concern. At a recent public County Council meeting, one resident expressed concern that 3 new coffee shops from a popular coffee shop chain were planning to open in the county, and the resident believed that this would create an increase in property crimes in the county. (Property crimes include burglary, larceny-theft, motor vehicle theft, and arson -- From [http://www.fbi.gov/about-us/cjis/ucr/crime-in-the-u.s/2010/crime-in-the-u.s.-2010/property-crime accessed on December 5, 2012].)

To support this claim, the resident presented the following data and scatterplot (with the least-squares line shown) for 8 counties in the state:

<table>
<thead>
<tr>
<th>County</th>
<th>Shops</th>
<th>Crimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>9</td>
<td>4000</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>2700</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>500</td>
</tr>
<tr>
<td>D</td>
<td>6</td>
<td>4200</td>
</tr>
<tr>
<td>E</td>
<td>15</td>
<td>6800</td>
</tr>
<tr>
<td>F</td>
<td>50</td>
<td>20800</td>
</tr>
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The scatterplot shows a positive linear relationship between "Shops" (the number of coffee shops of this coffee shop chain in the county) and "Crimes" (the number of annual property crimes for the county). In other words, counties with more of these coffee shops tend to have more property crimes annually.

a. Does the relationship between Shops and Crimes appear to be linear? Would you consider the relationship between Shops and Crimes to be strong, moderate, or weak?

b. Compute the correlation coefficient. Does the value of the correlation coefficient support your choice in part (a)? Explain.

c. The equation of the least-squares line for these data is:

$$\text{Predicted Crimes} = 1434 + 415.7(\text{Shops})$$

Based on this line, what is the estimated number of additional annual property crimes for a given county that has 3 more coffee shops than another county?

d. Do these data support the claim that building 3 additional coffee shops will necessarily cause an increase in property crimes? What other variables might explain the positive relationship between the number of coffee shops for this coffee shop chain and the number of annual property crimes for these counties?

e. If the following two counties were added to the data set, would you still consider using a line to model the relationship? If not, what other types (forms) of model would you consider?

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<td>24100</td>
</tr>
</tbody>
</table>

d. Example: Solution (DOK 3)
The scatterplot below shows the finishing times for the Olympic gold medalist in the men's 100-meter dash for many previous Olympic games. The line of best fit is also shown. (Source: http://trackandfield.about.com/od/sprintsandrelays/qt/olym100medals.htm)

![Olympic Gold Medalist - Men's 100-m Dash](image)

a. Is a linear model a good fit for the data? Explain, commenting on the strength and direction of the association.

b. The equation of the linear function that best fits the data (the line of best fit) is

\[ \text{Finishing time} = 10.878 - 0.0106 \times \text{(Year after 1900)}. \]

Given that the summer Olympic games only take place every four years, how should we expect the gold medalist's finishing time to change from one Olympic games to the next?

c. What is the vertical intercept of the function's graph? What does it mean in context of the 100-meter dash?

d. Note that the gold medalist finishing time for the 1940 Olympic games is not included in the scatterplot. Use the model to estimate the gold medalist's finishing time for that year.

e. What is a realistic domain for the linear function? Comment on how your answer pertains to using this function to make predictions about future Olympic 100-m dash race times.

8. Compute (using technology) and interpret the correlation coefficient of a linear fit.

(S-ID.C.8) (DOK 1,2)

a. Example: Solution (DOK 3)
Many counties in the United States are governed by a county council. At public county council meetings, county residents are usually allowed to bring up issues of concern. At a recent public County Council meeting, one resident expressed concern that 3 new coffee shops from a popular coffee shop chain were planning to open in the county, and the resident believed that this would create an increase in property crimes in the county. (Property crimes include burglary, larceny-theft, motor vehicle theft, and arson -- From http://www.fbi.gov/abou... accessed on December 5, 2012.)

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a. Does the relationship between Shops and Crimes appear to be linear? Would you consider the relationship between Shops and Crimes to be strong, moderate, or weak?

b. Compute the correlation coefficient. Does the value of the correlation coefficient support your choice in part (a)? Explain.

c. The equation of the least-squares line for these data is:

\[
\text{Predicted Crimes} = 1434 + 415.7(\text{Shops})
\]

Based on this line, what is the estimated number of additional annual property crimes for a given county that has 3 more coffee shops than another county?

d. Do these data support the claim that building 3 additional coffee shops will necessarily cause an increase in property crimes? What other variables might explain the positive relationship between the number of coffee shops for this coffee shop chain and the number of annual property crimes for these counties?

e. If the following two counties were added to the data set, would you still consider using a line to model the relationship? If not, what other types (forms) of model would you consider?

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9. **Distinguish between correlation and causation.** \(S\text{-ID.C.9}\) (DOK 1,2)
   a. Example: \textbf{Solution} (DOK 3)
Researchers have noticed that the number of golf courses and the number of divorcees in the United States are strongly correlated and both have been increasing over the last several decades. Can you conclude that the increasing number of golf courses is causing the number of divorcees to increase?

Either justify why a causation can be inferred, or explain what might account for the correlation other than a causal relation.

b. Example: Solution (DOK 3)
In a study of college freshmen, researchers found that students who watched TV for an hour or more on weeknights were significantly more likely to have high blood pressure, compared to those students who watched less than an hour of TV on weeknights. Does this mean that watching more TV raises one's blood pressure? Explain your reasoning.

c. Example: Solution (DOK 3)
Many counties in the United States are governed by a county council. At public county council meetings, county residents are usually allowed to bring up issues of concern. At a recent public County Council meeting, one resident expressed concern that 3 new coffee shops from a popular coffee shop chain were planning to open in the county, and the resident believed that this would create an increase in property crimes in the county. (Property crimes include burglary, larceny-theft, motor vehicle theft, and arson -- From http://www.fbi.gov/about-us/cjis/ucr/crime-in-the-u.s/2010/crime-in-the-u.s.-2010/property-crime accessed on December 5, 2012.)

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<td>24100</td>
</tr>
</tbody>
</table>

d. Example: Solution (DOK 3)
Ari has collected data on how much her classmates study each week and how well they did on their recent math test. Here is part of the data, showing the percentage of each group (divided according to hours spent studying) who got different grades on the test:

<table>
<thead>
<tr>
<th>Weekly Hours w Studying</th>
<th>C or below on test</th>
<th>B on test</th>
<th>A on test</th>
</tr>
</thead>
<tbody>
<tr>
<td>w &lt; 3</td>
<td>50%</td>
<td>20%</td>
<td>30%</td>
</tr>
<tr>
<td>3 ≤ w &lt; 6</td>
<td>60%</td>
<td>25%</td>
<td>15%</td>
</tr>
<tr>
<td>w ≥ 6</td>
<td>60%</td>
<td>20%</td>
<td>20%</td>
</tr>
</tbody>
</table>

a. What can you conclude about the percentage of Ari’s classmates who got B’s? What about the percent who got A’s?

b. Looking at the table, Ari decides that in order to do well in her math class, she should not study more than 3 hours a week: she has a 30 percent chance of getting an A as long as she does not study more than 3 hours per week. Is Ari’s reasoning valid?
Making Inferences and Justifying Conclusions

Understand and evaluate random processes underlying statistical experiments (S-IC.A)

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population. (S-IC.A.1) (DOK 1)
   a. Example: Solution (DOK 3)
      Students in a high school mathematics class decided that their term project would be a study of the strictness of the parents or guardians of students in the school. Their goal was to estimate the proportion of students in the school who thought of their parents or guardians as “strict”. They do not have time to interview all 1000 students in the school, so they plan to obtain data from a sample of students.

      a. Describe the parameter of interest and a statistic the students could use to estimate the parameter.

      b. Is the best design for this study a sample survey, an experiment, or an observational study? Explain your reasoning.

      c. The students quickly realized that, as there is no definition of “strict”, they could not simply ask a student, “Are your parents or guardians strict?” Write three questions that could provide objective data related to strictness.

      d. Describe an appropriate method for obtaining a sample of 100 students, based on your answer in part (a) above.

b. Example: Solution (DOK 3)
The 54 students in one of several middle school classrooms were asked two questions about musical preferences: “Do you like rock?” “Do you like rap?” The responses are summarized in the table below.

<table>
<thead>
<tr>
<th>Likes Rock</th>
<th>Doesn’t Like Rap</th>
<th>Row Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Likes Rock</td>
<td>27</td>
<td>6</td>
</tr>
<tr>
<td>Doesn’t Like Rock</td>
<td>4</td>
<td>17</td>
</tr>
<tr>
<td>Column Totals</td>
<td>31</td>
<td>23</td>
</tr>
</tbody>
</table>

a. Is this a random sample, one that fairly represents the opinions of all students in the middle school?

b. What percentage of the students in the classroom like rock?

c. Is there evidence in this sample of a positive association in this class between liking rock and liking rap? Justify your answer by pointing out a feature of the table that supports it.

d. Explain why the results for this classroom might not generalize to the entire middle school.

c. Example: Solution (DOK 3)

   From a class containing 12 girls and 10 boys, three students are to be selected to serve on a school advisory panel. Here are four different methods of making the selection.

   I. Select the first three names on the class roll.

   II. Select the first three students who volunteer.

   III. Place the names of the 22 students in a hat, mix them thoroughly, and select three names from the mix.

   IV. Select the first three students who show up for class tomorrow.

   Which is the best sampling method, among these four, if you want the school panel to represent a fair and representative view of the opinions of your class. Explain the weaknesses of the three you did not select as the best.

d. Example: Solution (DOK 3)

   For the 100 rectangles shown, the small squares that compose them have area 1, so the area of the rectangle is the number of squares, e.g., rectangle 7 has area 12.
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model? *(S-IC.A.2) (DOK 1,2)*
   a. Example: Solution (DOK 2)
A random sample of 100 students from a specific high school resulted in 45% of them favoring a plan to implement block scheduling. Is it plausible that a majority of the students in the school actually favor the block schedule? Simulation can help answer the questions.

The accompanying plot shows a simulated distribution of sample proportions for samples of size 100 from a population in which 50% of the students favor the plan, and another distribution from a population in which 60% of the students favor the plan. (Each simulation contains 200 runs.)

a. What do you conclude about the plausibility of a population proportion of 0.50 when the sample proportion is only 0.45?

b. What about the plausibility of 0.60 for the population proportion?

Population proportion 0.5; sample size 100

Population proportion 0.6; sample size 100

b. Example: Solution (DOK 3)
Many researchers have studied chimpanzees to learn about their problem solving skills. In 1978, researchers Premack and Woodruff published an article in *Science* magazine, reporting findings from a study on an adult chimpanzee named Sarah, who had been raised in captivity and had received extensive training using photos and symbols. In one experiment, Sarah was shown videotapes of eight different situations in which a human being was faced with a problem. After each videotape showing, Sarah was presented with two photographs, one of which depicted a possible solution to the problem. The researchers ensured that the order in which the photographs were presented was randomized (for example, the correct answer was not always presented first, etc.) and that the photographs had similar visuals (for example, similar colors, etc.) Of the eight problems, Sarah picked the photograph with the correct solution seven times. Could Sarah have been merely guessing and just lucky with her responses, or is there evidence that Sarah does better than just guessing?

a. Give two possible explanations for why Sarah might have answered seven out of eight correctly.

b. If Sarah were just guessing, and was just likely to pick one photograph compared to the other, how many would you expect her to get right out of eight problems?

c. Give an example of how you could use basic classroom tools (coins, dice, calculators, cards, etc.) to simulate one trial of Sarah “just guessing” to pick a photograph for one problem.
d. A student, James, decides to use simulation to investigate whether the study data provide evidence that Sarah was doing better than just randomly guessing, and so James tosses a coin eight times, and obtains six heads. Explain why James should repeat the process of tossing the coin eight times and recording the number of heads, many times.

e. James repeats the process of eight coin-tosses 100 times, each time recording the number of heads on the eight coin-tosses. The following is a dotplot of his results.

![Dotplot of coin-toss results]

Based on the above dotplot, what was the most common result for "number of heads" in eight coin-tosses? Why does this make sense?

f. Based on this dotplot, would you say that a score of 7 out of 8 would be unusual if Sarah has just been guessing? Why or why not?

g. Which of the following is a possible explanation for Sarah's performance?

   i. Sarah had been just guessing and got lucky with her responses.
   ii. Sarah does better than just guessing,
   iii. Both (i) and (ii) are possible explanations.

h. Based on the simulation results, which of the following appears to be a plausible (likely) explanation for Sarah's performance?

   i. Sarah had been just guessing and got lucky with her responses.
   ii. Sarah does better than just guessing,
   iii. Both (i) and (ii) are possible explanations.

i. Based on the results of this study, would it be reasonable to say that all chimpanzees do better than just guessing when faced with the kind of problems posed to Sarah? Explain why or why not.
c. Example: Solution (DOK 3)

An instructor in a classroom of 40 students gave each of the 40 students a bag of 10 chips. In each bag, some of the 10 chips were red, and the rest were white. The bags were made of a thick cloth so that no one could see inside the bags, and all of the bags were identical in terms of their contents.

Each student was asked to remove one chip from his or her bag and to raise his or her hand if that chip was red. (This constitutes one “round” of the game.) Those who selected a red chip were allowed to remain in the game and play another round; those who selected a white chip were “out” of the game. Each student who was still in the game was asked to place his/her chip back into the bag and to repeat the selection process when asked. This continued until there was only one student left.

1) If you were told that each bag contained 6 red chips and 4 white chips, what is the expected number of students who would raise their hands in the first round of this game?

2) Assuming that each bag contained 6 red chips and 4 white chips, fill in the table below indicating how many students you would expect to raise a hand in each round. (Note: it is ok to have a non-integer expected number of students in any round of the models you are developing.)
3) The following represent the actual results from the classroom.

<table>
<thead>
<tr>
<th>Round</th>
<th># of students still in game</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>40</td>
</tr>
<tr>
<td>1</td>
<td>26</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

At the end of the game, the instructor asked “Knowing that each bag has 10 chips, and knowing that everyone’s bags are the same, how many red chips do you believe are in each bag?”
a) Develop a table similar to the one you made in part 2 above based on the assumption of 5 red chips in a bag. Make a second table based on the assumption of 7 red chips in a bag.

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- Expected values based on 5 (out of 10) red chips in each bag.

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<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
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<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
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- Expected values based on 7 (out of 10) red chips in each bag.

b) Based on the actual results and the 3 models you have developed, do you believe that each bag had 5, 6, or 7 red chips? Choose one value and explain.

d. Example: **Solution** (DOK 3)

An instructor in a classroom of 40 students gave each of the 40 students a bag of 10 chips. In each bag, some of the 10 chips were red, and the rest were white. The bags were made of a thick cloth so that no one could see inside the bags, and all of the bags were identical in terms of their contents.

Each student was asked to remove one chip from his or her bag and to raise his or her hand if that chip was red. The number of reds for the entire class was recorded for that iteration. All students then placed their chips back in their bags, and the process was repeated 31 more times. The classroom results, a summary table, and a dotplot summarizing the results are as follows:
1) Below are 4 histograms that show the expected probability distribution of the variable "number of reds in 40 trials" based on 4 different probability assumptions regarding the red chips. (Note: these graphs are generated based on something called a **binomial distribution**.)
Using the class results as shown in the dotplot as a guide, compare the dotplot to these probability histograms and determine which of the 4 values for \( P(\text{Red}) \) (i.e., .40, .50, .60, or .70) was most likely the correct probability for the bags the instructor used in her class. Explain your choice of probability by commenting on the graphical features of these distributions, specifically mentioning those features that “fit” your choice of probability and those graphical features that eliminated other choices for you.

Make inferences and justify conclusions from sample surveys, experiments, and observational studies (S-IC.B)

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. (S-IC.B.3) (DOK 1,2)
   a. Example: Solution (DOK 3)

   Students in a high school mathematics class decided that their term project would be a study of the strictness of the parents or guardians of students in the school. Their goal was to estimate the proportion of students in the school who thought of their parents or guardians as “strict”. They do not have time to interview all 1000 students in the school, so they plan to obtain data from a sample of students.

   a. Describe the parameter of interest and a statistic the students could use to estimate the parameter.

   b. Is the best design for this study a sample survey, an experiment, or an observational study? Explain your reasoning.

   c. The students quickly realized that, as there is no definition of “strict”, they could not simply ask a student, “Are your parents or guardians strict?” Write three questions that could provide objective data related to strictness.

   d. Describe an appropriate method for obtaining a sample of 100 students, based on your answer in part (a) above.

   b. Example: Solution (DOK 3)
A student interested in comparing the effect of different types of music on short-term memory conducted the following study: 80 volunteers were randomly assigned to one of two groups. The first group was given five minutes to memorize a list of words while listening to rap music. The second group was given the same task while listening to classical music. The number of words correctly recalled by each individual was then measured, and the results for the two groups were compared.

a. Is this an experiment or an observational study? Justify your answer.

b. In the context of this study, explain why it is important that the subjects were randomly assigned to the two experimental groups (rap music and classical music).

c. Example: Solution (DOK 3)

In a study of college freshmen, researchers found that students who watched TV for an hour or more on weeknights were significantly more likely to have high blood pressure, compared to those students who watched less than an hour of TV on weeknights. Does this mean that watching more TV raises one’s blood pressure? Explain your reasoning.

d. Example: Solution (DOK 3)

The following are some common methods of data collection in statistical studies that involve people.

Broadly speaking, a survey is a way of learning about a group of people by having some of the people in the group answer questions. A survey might involve completing a form, completing a questionnaire online, participating in a personal interview, or answering questions over the phone.

An observational study is a type of study where a researcher uses observed information to learn about a group of individuals. The individuals being observed are specifically not interfered with aside from the measuring of their responses. For example, if we wanted to learn if a person’s hair color is related to his/her favorite soda, we would record values of the variables “hair color” and “favorite soda” from observation or from reported observations, such as through a survey. (Surveys are one type of observational study.)

A sample survey is a survey that is carried out using a sample of people who are intended to represent a larger population. For example, if we wanted to know more about the opinions of the one million adults who live in a particular city, and if we selected a sample of 900 individuals from those one million people, a survey conducted using those 900 people would be considered a sample survey. Well designed sample surveys can provide data that can be used in a variety of ways. For example, a survey might be used to assess the views of the general public, to examine trends in customer behavior, or to estimate the values of population characteristics when a census of an entire population is not practical.
An experiment is different from an observational study in that a researcher deliberately imposes different treatments on different groups and then compares the groups to determine if there is any difference in a specific variable of interest (called a response). A well designed experiment allows researchers to decide if the different treatments result in different responses.

1) Using the descriptions above, determine if each study described below is a sample survey, an observational study, or an experiment.

a) A pharmaceutical company is interested in comparing three brands of pain relievers. The company wants to know if one brand is much better than the other brands in terms of mobility improvement. Three different groups of 30 people participate in the study. Each group gets one of the three brands of pain relievers. The improvement in mobility of these 90 people is recorded after 5 weeks of taking the medicine.

b) Medical records of a group of long-time residents of a town are compared to the medical records of a group of long-time residents of another town. Researchers are curious to see if one town appears to be “healthier” in general.

c) A group of 600 registered voters in a given county are asked how they intend to vote in an upcoming election. A summary of their responses is posted on a news web-site, and it is implied that this group is representative of all registered voters in that county. The responses are used to predict the outcome of the election.

d) In a study to see if administering a medication orally or by injection makes a difference in how quickly people with back pain feel relief, a group of 400 adults suffering from back pain is divided into two groups. One group of 200 people is given the medication in pill form, while the other group of 200 people is given the medication in the form of a shot. Results for each group are compared to see if there is a difference in pain relief between the two methods.
Why Randomization Matters

Randomization plays an important part in data collection in two distinct ways:

Random Selection – In the case of a sample survey or observational study, we want the sample of people involved to be representative of their greater population. To help achieve this, we want the people to be impartially selected so that we do not introduce any favoritism in the selection which could potentially distort conclusions drawn from the data.

Random Assignment – In the case of an experiment, we want to impartially assign the people (called subjects) to the various treatment groups. If we do not do this, we might inadvertently create a situation where any observed differences between our groups may have been caused by some factor other than the treatments we are interested in.

2) The four studies mentioned in Question 1 are presented here again. However, additional information is now provided regarding how the people involved were chosen or assigned. Each of these studies contains a flaw in terms of how the people involved were chosen or assigned. For each study, explain why the data collection method might cause a problem, and how randomization should be properly employed.
a) A pharmaceutical company is interested in comparing three brands of pain relievers. The company wants to know if one brand is much better than the other brands in terms of mobility improvement. Three different groups of 30 people participate in the study. Each group gets one of the three brands of pain relievers. The improvement in mobility of these 90 people is recorded after 5 weeks of taking the medicine. **The first group of 30 patients were patients of Dr. Smith, the second group of 30 patients came from Dr. Jones, and the last group of 30 came from Dr. McGillicuddy.**

b) Medical records of a group of long-time residents of a town are compared to the medical records of a group of long-time residents of another town. Researchers are curious to see if one town appears to be “healthier” in general. **In order to save time and money, all of the records were selected from the largest of the four hospitals in Town X and from the largest of the three hospitals in Town Y.**

c) A group of 600 registered voters in a given county are asked how they intend to vote in an upcoming election. A summary of their responses is posted on a news web-site, and it is implied that this group is representative of all registered voters in that county. The responses are used to predict the outcome of the election. **The 600 voters submitted their opinions via a webpage and were able to choose to participate after seeing an advertisement on television.**

d) In a study to see if administering a medication orally or by injection makes a difference in how quickly people with back pain feel relief, a group of 400 adults suffering from back pain is divided into two groups. One group of 200 people is given the medication in pill form, while the other group of 200 people is given the medication in the form of a shot. Results for each group are compared to see if there is a difference in pain relief between the two methods. **People were allowed to select whether they would take the medicine orally or by shot. Once 200 people had filled up one of the groups, all remaining people were immediately placed in the other group.**
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. \((S-I.C.B.4)\) [DOK 2]
   a. Example: Solution [DOK 3]

Sometimes hotels, malls, banks, and other businesses will present a display of a large, clear container holding a large number of items and ask customers to estimate some aspect of the items in the container as a contest. In some cases, contestants are allowed to sample items from the jar; but usually, contestants simply have to estimate based on visual inspection of the jar. A local bank is running such a contest, but one of the bank employees is concerned.

The bank has placed 1,500 marbles in a very large, clear jar near the customer entrance. Since the bank's logo's colors are blue and white, some of the 1,500 marbles are blue and the rest are white. In order to enter the contest, a customer must fill in an entry form with his/her estimate for the percentage of blue marbles in the jar and then place the entry form in a ballot box. A random drawing will be held and the first entry drawn that correctly estimates the percentage of blue marbles in the whole jar will receive a $100 gift certificate. The entry form says the following:

   Name: ________________  Phone: ________________

   I think that 1 out of every ______ marbles in this jar is blue.

   (Fill in the blank with a "2", "3", "4", "5", or "6").

Note that for the ease of the contestants, the estimate is to be stated as "1 out of every 2" instead of "50%," "1 out of every 3" instead of "33.3%," and so on.
Now the concerned employee is fairly confident that the true proportion of blue marbles is 25% (1 out of every 4), but he has heard other employees (some of whom are responsible for the contest) state a true proportion value that is different. The employee is worried enough that he wants to investigate but he certainly does not want to empty the jar and inspect all 1,500 marbles! He decides to select a random sample of marbles from the jar and calculate the percentage of blue marbles in his sample. The percentage of blue marbles in the random sample will be his estimate for the actual percentage of blue marbles in the jar.

He selects a random sample of 5 marbles, and only 1 of the marbles is blue. Based on this sample which gives him an estimate of 20% (1 out of 5) blue marbles, the employee is concerned, but he decides to stick with his original claim of 25% blue marbles in the jar. However, he is now inspired to take even larger samples. He records his results in the table below (additional spaces will be filled in eventually).
<table>
<thead>
<tr>
<th>Sample Number</th>
<th>Sample Size</th>
<th>Total Number of Blue Marbles in Sample</th>
<th>Percentage of Blue Marbles in Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
<td>$\frac{1}{5} = 20%$</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>2</td>
<td>$\frac{2}{12} = \frac{1}{6}$</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>32</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
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<td></td>
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<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. His second random sample consists of 12 marbles. Only 2 of the marbles are blue. (This is recorded in the table above.) Compute the sample percentage of blue marbles for this sample and record it in the table. Based on this sample, do you think the employee should stick with his original claim of 25% blue marbles in the jar, or should he come up with a different estimate? Explain why you think this.

b. He then takes a random sample of 20 marbles (Sample 3). Five of the 20 marbles are blue. Compute the sample percentage of blue marbles for this sample and record it in the table. Based on this sample, do you think the employee should stick with his original claim of 25% blue marbles in the jar, or should he come up with a different estimate? Explain why you think this.
c. He then takes a random sample of 32 marbles (Sample 4). Eight of the marbles are blue. Enter this information on the table, and compute the sample percentage of blue marbles for this sample. Based on this sample, do you think the employee should stick with his original claim of 25% blue marbles in the jar, or should he come up with a different estimate? Explain why you think this.

At this point, the employee feels compelled to talk to the bank manager who is responsible for the contest. The bank manager is a little surprised by the results, but she is not overly concerned. She is quite confident that the true proportion of blue marbles is 33.3%, or 1 in every 3 (i.e., 5,000 blue, 10,000 white marbles), and she asks the concerned employee to go back and look at an even larger random sample of marbles.

d. He then takes a random sample of 40 marbles (Sample 5) and 13 of the marbles are blue. Add this information to the table. Based on this sample, and mindful that the correct, true proportion of blue marbles in the jar is 1 in 2, or 1 in 3, or 1 in 4, etc., do you think the employee should challenge the bank manager's claim that 1 in every 3 marbles is blue? Explain why you think this.

e. Here are the results of some additional random samples. Record each of these in the table and compute the blue marble percentage for each sample.
Sample 6, sample size = 55, 17 blue.
Sample 7, sample size = 65, 21 blue.
Sample 8, sample size = 75, 24 blue.
Sample 9, sample size = 85, 27 blue.

f. Based on the random sample of 85 marbles, and mindful that the correct, true proportion of blue marbles in the jar is 1 in 2, or 1 in 3, or 1 in 4, etc., do you think that the employee should challenge the bank manager’s claim that 1 in every 3 marbles is blue? Explain why you think this.

g. Keeping in mind that the samples were random samples, and assuming that the bank manager’s claim is correct that the true proportion of blue marbles is 33.3% (1 in every 3), did the employee get more accurate estimates from the small samples or from the large samples?

b. Example: Solution (DOK 3)

Background:

Researchers have questioned whether the traditional value of 98.6°F is correct for a typical body temperature for healthy adults. Suppose that you plan to estimate mean body temperature by recording the temperatures of the people in a random sample of 10 healthy adults and calculating the sample mean. How accurate can you expect that estimate to be? In this activity, you will develop a margin of error that will help you to answer this question.

Let’s assume for now that body temperature for healthy adults follows a normal distribution with mean 98.6 degrees and standard deviation 0.7 degrees. Here are the body temperatures for one random sample of 10 healthy adults from this population:

<table>
<thead>
<tr>
<th>Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>97.73</td>
</tr>
<tr>
<td>98.76</td>
</tr>
<tr>
<td>98.27</td>
</tr>
<tr>
<td>99.95</td>
</tr>
<tr>
<td>98.47</td>
</tr>
<tr>
<td>98.49</td>
</tr>
<tr>
<td>98.97</td>
</tr>
<tr>
<td>98.68</td>
</tr>
<tr>
<td>99.27</td>
</tr>
<tr>
<td>99.25</td>
</tr>
</tbody>
</table>

a. What is the mean temperature for this sample?

b. If you were to take a different random sample of size 10, would you expect to get the same value for the sample mean? Explain.

Below is a dot plot of the sample mean body temperature for 100 different random samples of size 10 from a population where the mean temperature is 98.6 degrees.
c. How many of the samples had sample means that were greater than 98.5 degrees and less than 98.7 degrees?

d. Based on the dot plot above, if you were to take a different random sample from the population, would you be surprised if you got a sample mean of 98.8 or greater? Explain why or why not.

e. Which of the following statements is appropriate based on the dot plot of sample means above?

Statement 1: Most random samples of size 10 from the population would result in a sample mean that is within 0.1 degrees of the value of the population mean (98.6).

Statement 2: Most random samples of size 10 from the population would result in a sample mean that is within 0.3 degrees of the value of the population mean (98.6).

Statement 3: Most random samples of size 10 from the population would result in a sample mean that is within 0.5 degrees of the value of the population mean (98.6).

The margin of error associated with an estimate of a population mean can be interpreted as the maximum likely difference between the estimate and the actual value of the population mean for a given sample size.

f. Explain why 0.45 degrees would be a reasonable estimate of the margin of error when using the sample mean from a random sample of size 10 to estimate the mean body temperature for the population described above.

g. If you were to use a random sample of size 20 to estimate the population mean, would you expect the estimate to be closer to or farther from the actual value of the population mean than if you had used a random sample of size 10? Would this mean that the margin of error would be less than or greater than the margin of error for a sample of size 10?

h. Below is a dot plot of the sample mean temperature for 100 different random samples of size 20 from a population with an actual mean temperature of 98.6 degrees. Explain how this dot plot supports your answer in part (g).
Below is a comparative dot plot that shows sample means for 100 random samples for each of the sample sizes 10, 20, 40, and 100.

i. What is a reasonable estimate of the margin of error for samples of size 20? For samples of size 40? For samples of size 100?

j. How is the margin of error related to sample size?

In practice, we don’t take many random samples from a population and we don’t know the actual value of the population mean, so we need a way to estimate the margin of error from a single sample.

An estimate of the margin of error based on a single random sample can be obtained by evaluating the following expression

\[ \text{estimated margin of error} = \frac{s}{\sqrt{n}} \]

where \( s \) is the sample standard deviation and \( n \) is the sample size.

k. Using the sample at the beginning of this activity, what is the estimated margin of error?

l. Suppose that a random sample of 50 healthy adults resulted in a sample mean body temperature of \( \bar{x} = 98.2 \) degrees and a sample standard deviation of \( s = 0.65 \) degrees. Would you consider this evidence that the actual mean temperature for healthy adults is in fact less than 98.6 degrees? (Hint: what is the estimated margin of error?)

c. **Example: Solution** (DOK 3)
Roadside flares are often used by motorists to warn oncoming drivers of obstacles in the roadway and to draw attention to hazardous road conditions. Generally, flares are small and portable. One of the great conveniences of the flares is that they do not require electricity. The light from the flare is caused by a chemical reaction from elements inside the flare, and they can be used in many conditions.

Fred’s Flare Company made flares that burned for 100 minutes on average. However, Fred has developed a new chemical process that should allow the flares to now burn slightly over 120 minutes (that is, over 2 hours) on average; and the claim of “Now burns for over 2 hours on average” will be stated on the packages in which the new flares are sold.

Because of this claim on the package, the company must periodically sample a group of flares and check to see if the “Now burns for over 2 hours on average” claim is reasonable for the population of all flares they manufacture.

1. If every manufactured flare was sampled, that would be called a census, and the population average burn time could be computed directly. Give at least one reason why a census would not be feasible in this case.

**Random Sampling**

If random sampling is used, it is reasonable for Fred to use the mean burning time of a given sample (a sample mean) to estimate the mean burning time of his entire population (the population mean). Knowing that there is variability in all manufacturing, Fred knows that even if the mean burn time of all of the flares is actually over 120 minutes on average, he could occasionally get a sample with an average burn time that is not over 120 minutes. However, he also wants to be confident that a population mean burning time of over 120 minutes is plausible.

Even though Fred’s company will manufacture several thousand flares each day, sampling is somewhat costly, so Fred wants to sample as few flares as possible. Previously, when the old “100 minute” flares were tested, the company would take a sample of 20 flares each hour from the production line and compute a mean burning time for each sample of 20 flares. Fred determines that this sampling method should continue for the new flares. The results of one full day of this hourly testing are shown below. (One day = 24 hourly samples, so 24 sample means shown in the dotplot.)

![Dotplot](image)

2. According to the dotplot, how many of these 24 sample averages are at or below 120 minutes? What percentage of these 24 sample averages are at or below 120 minutes? What percentage of the samples are strictly below 120 minutes?
3. By visual inspection of the dotplot, estimate the values in the 5-number summary for these 24 hourly sample averages. What is the range of these 24 sample average measurements?

The big question for Fred is: do the results of this day’s sampling raise concern about the “Now burns for over 2 hours on average” claim?

4. Use the dotplot and/or the analysis you’ve performed above to address Fred’s question. Be thorough and mention any information that seems to support the “Now burns for over 2 hours on average” claim and any information that would not encourage you to support that claim. In one sentence, what will you tell Fred? Are you confident that the population mean burning time is more than 120 minutes?

A Larger Sample Size

Fred is concerned by the results, but he is still fairly sure that the population average burning time is slightly more than 120. He decides to sample 100 flares every hour instead of just 20. A dotplot showing the averages from the 24 samples of size 100 from this second day of sampling is as follows:

Keep in mind that the MANUFACTURING PROCESS DID NOT CHANGE, only the sample size for the hourly sampling was changed -- specifically it was increased to 5 times its original size (from 20 flares to 100 flares per sample).

Now re-examine the previous questions using this new dotplot.

5. According to this NEW dotplot, how many of the 24 sample averages are at or below 120 minutes?

6. By visual inspection of the dotplot, estimate the values in the 5-number summary for these 24 NEW sample averages. What is the range of these 24 sample averages?

7. Using this NEW dotplot that came from sampling that used a larger sample size and the analysis you’ve just performed, what general information about these 24 sample averages seems to support the claim that the population mean burning time is more than 120 minutes?

Comparing the Dotplots and the Sample Sizes

8. Which distribution of the 24 hourly averages had the smaller range: the distribution of sample averages based on a sample of size 20, or the distribution of sample averages based on a sample of size 100?
To further examine the effect of sample size, consider the following histograms representing two sampling simulations from the same flare population. In the first simulation, we imagine that Fred continued to take random samples of size 20 every hour for 30 days. In the second simulation, we imagine that Fred continued to take random samples of size 100 every hour for 30 days. Note: both distributions represent the averages from 720 samples (720 = 24 hours * 30 days).

9. Assuming that the cost to sample 100 flares per hour is only slightly more than the cost to sample 20 flares per hour, would you recommend using a larger sample size or a smaller sample size to estimate the population mean? Explain.
Margin of Error

In the dotplots given earlier, each dot represented a sample mean and each sample mean is an estimate of the population mean burning time. When random sampling is used, in the long run, sample means generated from many random samples tend to be centered around the actual population mean. For example, here are the averages of the four distributions shown above:

When the sample was size 20, the average of the 24 hourly sample means = 120.46 minutes

When the sample was size 100, the average of the 24 hourly sample means = 120.49 minutes

When the sample was size 20, the average of the 720 hourly sample means = 120.50 minutes

When the sample was size 100, the average of the 720 hourly sample means = 120.50 minutes

In other words, the “average of all the sample averages” in all 4 cases was about 120.5 minutes (rounded). That would be a good estimate for the population mean burning time. Unfortunately, in most cases, you don’t collect many random samples—you only select one random sample for analysis.

A margin of error is loosely defined as the largest expected size of the difference between an estimate and the actual population value that is being estimated. For example, if you were trying to estimate the average weight of a population based on a proper sample and your margin of error was stated as “2 pounds,” that is saying that you would be very confident that the actual population mean weight is within 2 pounds of your sample estimate. In other words, if you obtained a sample average weight of 45 pounds and your margin of error was 2 pounds, you would be very confident that the actual population mean weight would be somewhere between 43 pounds (that’s 45 – 2 pounds) and 47 pounds (that’s 45 + 2 pounds).

One informal way of developing a margin of error from a simulation is to compute the midrange (= range/2) of the simulations results. For the first simulation histogram (the one based on samples of size n = 20), this informal margin of error value would be roughly 1.8 minutes (you can confirm this above). If the true population mean was in fact 120.5 minutes, that value would be within the margin of error of every one of the 720 estimates. In other words, no matter which one of the 720 estimates you selected, the value “120.5 minutes” would be within 1.8 minutes of that estimate.

10. If we perform this same informal method of developing a margin of error using the second simulation histogram (the one based on samples of size n = 100), what would the value for the margin of error be?
11. Generally speaking, do you think that when proper random sampling occurs, the margin of error for estimating a population mean gets smaller or larger as the sample size increases?

d. Example: **Solution** (DOK 3)

Many large retail stores and restaurants offer special discounts and free gifts to customers throughout the year. In some cases (particularly fast-food restaurants), some form of “instant win” contest takes place where a customer earns a ticket with a particular food purchase. The ticket states that the customer either has won a food/beverage prize with the ticket (for example, “Instant Win: Free 12 oz. soda!”) or has not won anything with the ticket (for example, “Sorry, you are not a winner. Please play again soon.”)

A certain fast-food restaurant chain offers such a contest in a “scratch ‘n win” format where customers must scratch off a silver coating on a ticket to reveal the outcome. In its commercials, the restaurant chain says that “1 in 5 tickets is a Winner.” However, due to some unfortunate printing and shipping errors, there is now concern that fewer winning tickets have been distributed than originally planned and that the true proportion of winners is in fact less than the 0.20 that was claimed by the “1 in 5 tickets is a Winner” statement. The restaurant chain’s management does not want to get in trouble with the public and be accused of fraud, so they decide to perform some sampling to see if a winning proportion of 0.20 is plausible for the population of tickets that were distributed. They decide to ask the local owners at each of the chain’s 35 most popular locations to conduct some sampling.

**Random Sampling**

If random sampling is used, it is reasonable for the restaurant management to use the proportion of winning tickets in a given sample (a **sample proportion**) to estimate the proportion of winning tickets in the entire population (the **population proportion**). For now, we will assume that the population of tickets is extremely large, that the local owners will use a randomization method to select the customers to approach, and that the customers who are approached about their tickets will answer truthfully.

Management knows that even if the population proportion of winning tickets is actually 0.20 on occasion, the sample proportions will have values that are less than 0.20 due to natural sampling variability. However, they also want to be confident that a population proportion of 0.20 for the winning tickets is plausible.

The results of one day of this sampling are shown below. There are 35 observations, and each observation (dot) represents the sample proportion obtained from a given restaurant based on a random sample of 28 tickets at the restaurant. (Each restaurant randomly selected 4 tickets per hour for 7 hours.)

![Sample Proportions](image-url)
1. None of the sample proportions appear to be exactly equal to 0.20. Explain why obtaining a sample proportion of 0.20 is not possible when using a sample of size 28.

2. According to the dotplot, how many of these 35 sample proportions are below 0.20? Based on these 35 observations, what is the probability that a randomly selected location's sample proportion was below 0.20?

3. By visual inspection of the dotplot distribution, estimate the values in the 5-number summary for these 35 sample proportions and comment on the shape, center, and range of the distribution.

The important questions for the restaurant chain's management are: do the results of today's sampling raise concern about the "1 in 5 tickets is a Winner" claim? Could the actual population proportion of winning tickets actually be less than 0.20?

4. Use the dotplot and/or the analysis you've performed above to address that question. Be thorough and mention any information that would encourage you to dismiss the "1 in 5 tickets is a Winner" claim. Do you feel there is enough evidence to challenge the "1 in 5 tickets is a Winner" claim, or do you feel that the claim should be "left alone" and not disputed?

A Larger Sample Size

Even though the sampling accounted for 980 total tickets (980 = 28 tickets each * 35 locations), some of the company's executives were concerned that only 28 tickets were sampled at each restaurant. In an urgent memo, management now decides to ask their local owners to now sample 345 tickets in each of the 35 locations (that's about 49 or 50 tickets per hour for 7 hours). A dotplot showing the sample proportions from the 35 restaurant locations on this second day of sampling (when the sample size of 345 tickets was used at each location) is as follows:

Keep in mind that the population proportion of winning tickets DID NOT CHANGE, only the sample size for each restaurant's sampling was changed -- specifically it was increased to over 12 times its original size (from 28 tickets to 345 tickets per sample).

Now re-examine the previous questions using this new dotplot.

5. According to this NEW dotplot, how many of these 35 sample proportions are below 0.20?
6. By visual inspection of the dotplot, estimate the values in the 5-number summary for these 35 NEW sample proportions. What is the range of these 35 sample proportions?

7. Using this NEW dotplot, what general information about these 35 sample proportions seems to support the claim that the population proportion of winning tickets is less than 0.20? Do you feel there is enough evidence to challenge the “1 in 5 tickets is a Winner” claim, or do you feel that the claim should be “left alone” and not disputed?

**Comparing the Dotplots and the Sample Sizes**

8. Which distribution of sample proportions had the smaller range: the distribution based on a sample of size \(n = 28\) or the distribution based samples of size \(n = 345\)?

To further examine the effect of sample size, consider the following histograms representing two sampling simulations **from the same ticket population**. In the first simulation, we imagine that the restaurant management has asked 1000 of its restaurants to randomly sample 28 tickets on a given day. In the second simulation, we imagine that the restaurant management has asked 1000 of its restaurants to randomly sample 345 tickets on a given day. Note: both distributions represent the sample proportions from 1000 random samples.
9. Since the local owners who are performing the sampling would pretty much follow whatever instructions they received from restaurant management, would you recommend using a larger sample size or a smaller sample size to estimate the population proportion? Explain.
Margin of Error

In the dotplots given earlier, each dot represented a sample proportion; and each sample proportion is an estimate of the population proportion of winning tickets. When random sampling is used, in the long run, sample proportions generated from many random samples tend to be centered around the actual population proportion. For example, here are the averages of the four distributions shown above:

When the sample was size 28, the average of the 35 sample proportions = 0.189

When the sample was size 345, the average of the 35 sample proportions = 0.169

When the sample was size 28, the average of the 1000 sample proportions = 0.172

When the sample was size 345, the average of the 1000 sample proportions = 0.171

Notice that these four averages are about the same, indicating that the distributions of the sample proportions were all centered at about the same place. Also notice that the “average of all the sample proportions” in all 4 cases was below 0.20; and in 3 of the cases, this average value was quite close to 0.17 -- and that would be a good estimate for the population proportion of winning tickets. Unfortunately, in most analyses, you don't get to collect many random samples as was done here — you only get to select one random sample for your analysis.
A margin of error is loosely defined as the largest expected size of the difference between an estimate and the actual population value that is being estimated. For example, if you were trying to estimate the population proportion of voters who supported a political candidate and your margin of error was stated as "0.03," that is saying that you would be very confident that the actual population proportion of voters who supported the political candidate is within 0.03 of your sample estimate. In other words, if you obtained a sample proportion of 0.45 and your margin of error was 0.03, you would be very confident that the actual population proportion would be somewhere between 0.42 (that's 0.45 – 0.03) and 0.48 (that's 0.45 + 0.03).

One informal way of developing a margin of error from a simulation is to compute the value that is the range of the simulation's results divided by 2 (margin of error = range/2). For the first simulation histogram (the one based on samples of size n = 28), this informal margin of error value would be roughly 0.20 (you can confirm this above). If the true population proportion was in fact 0.17, that value would be within the margin of error of nearly every one of the 1000 estimates. In other words, the value “0.17” would be within plus or minus 0.20 of almost any of the 1000 estimates in the histogram. (In fact, “0.17” is within plus or minus 0.20 of 997 of the 1000 estimates.)

10. If we perform this same informal method of developing a margin of error using the second simulation histogram (the one based on samples of size n = 345), what would the value for the margin of error be approximately? Using this margin of error value, how many of the estimates in the dotplot from Questions 5 – 7 are within this margin of error of the hypothesized population proportion value of 0.20?

11. Generally speaking, do you think that when proper random sampling occurs, the margin of error for estimating a population proportion gets smaller or larger as the sample size increases?

5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. *(S-IC.B.5) (DOK 2,3)*

6. Evaluate reports based on data. *(S-IC.B.6) (DOK 2,3)*
Conditional Probability and the Rules of Probability

Understand independence and conditional probability and use them to interpret data (S-CP.A)

Example: At a local fair, the price of admission includes the opportunity for a person to spin a wheel for free ride tickets.

- Each spin of the wheel is a random event.
- The result from each spin of the wheel is independent of the results of previous spins.
- Each spin of the wheel awards tickets according to the probabilities shown below.

<table>
<thead>
<tr>
<th>Spin the Wheel</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ticket</td>
</tr>
<tr>
<td>2 tickets</td>
</tr>
<tr>
<td>3 tickets</td>
</tr>
<tr>
<td>5 tickets</td>
</tr>
<tr>
<td>10 tickets</td>
</tr>
</tbody>
</table>

Let $X$ be the number of tickets a person wins based on 2 spins. There are 13 possible values for $X$.

Some values of $X$ are more common than others. For example, winning only 2 tickets in 2 spins is a somewhat common occurrence with probability 0.1225. It means the person wins 1 ticket on the first spin and 1 ticket on the second spin ($0.35 \times 0.35$).

A list of the possible values of $X$ and the corresponding probabilities for most values of $X$ is shown below.

Fill in the three missing probability values in the table.

<table>
<thead>
<tr>
<th>$X$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1225</td>
</tr>
<tr>
<td>3</td>
<td>0.1750</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.1000</td>
</tr>
<tr>
<td>6</td>
<td>0.1450</td>
</tr>
<tr>
<td>7</td>
<td>0.0750</td>
</tr>
<tr>
<td>8</td>
<td>0.0600</td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.0350</td>
</tr>
<tr>
<td>12</td>
<td>0.0250</td>
</tr>
<tr>
<td>13</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.0150</td>
</tr>
<tr>
<td>20</td>
<td>0.0025</td>
</tr>
</tbody>
</table>
1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not") (S-CP.A.1) (DOK 1,2)
   a. Example: Solution (DOK 3)
On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Data on survival of passengers are summarized in the table below. (Data source: http://www.encyclopedia-titanica.org/titanic-statistics.html)

<table>
<thead>
<tr>
<th></th>
<th>Survived</th>
<th>Did not survive</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>First class passengers</td>
<td>201</td>
<td>123</td>
<td>324</td>
</tr>
<tr>
<td>Second class passengers</td>
<td>118</td>
<td>166</td>
<td>284</td>
</tr>
<tr>
<td>Third class passengers</td>
<td>181</td>
<td>528</td>
<td>709</td>
</tr>
<tr>
<td>Total passengers</td>
<td>500</td>
<td>817</td>
<td>1317</td>
</tr>
</tbody>
</table>

a. Calculate the following probabilities. Round your answers to three decimal places.

i. If one of the passengers is randomly selected, what is the probability that this passenger was in first class?

ii. If one of the passengers is randomly selected, what is the probability that this passenger survived?

iii. If one of the passengers is randomly selected, what is the probability that this passenger was in first class and survived?

iv. If one of the passengers is randomly selected from the first class passengers, what is the probability that this passenger survived? (That is, what is the probability that the passenger survived, given that this passenger was in first class?)

v. If one of the passengers who survived is randomly selected, what is the probability that this passenger was in first class?

vi. If one of the passengers who survived is randomly selected, what is the probability that this passenger was in third class?

b. Why is the answer to part (a.iv) larger than the answer to part (a.iii)?

c. Why is the answer to part (a.v) larger than the answer to part (a.vi)?

d. What other questions can you ask and answer using information in the given table? List at least three.

b. Example: **Solution** (DOK 3)
In order to play a popular “spinning wheel” game at Fred's Fun Factory Arcade, a player is required to pay a small, fixed amount of 25 cents each time he/she wants to make the wheel spin. When the wheel stops, the player is awarded tickets based on where the wheel stops -- and these tickets are then redeemable for prizes at a redemption center within the arcade.

This particular game has no skill component; each spin of the wheel is a random event, and the results from each spin of the wheel are independent of the results of previous spins.

The wheel awards tickets with the following probabilities:

<table>
<thead>
<tr>
<th>Tickets</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ticket</td>
<td>35%</td>
</tr>
<tr>
<td>2 tickets</td>
<td>20%</td>
</tr>
<tr>
<td>3 tickets</td>
<td>20%</td>
</tr>
<tr>
<td>5 tickets</td>
<td>10%</td>
</tr>
<tr>
<td>10 tickets</td>
<td>10%</td>
</tr>
<tr>
<td>25 tickets</td>
<td>4%</td>
</tr>
<tr>
<td>100 tickets</td>
<td>1%</td>
</tr>
</tbody>
</table>

A young girl is given 2 quarters so that she can play the game two times. Let $X$ be the number of tickets she wins based on two spins. There are 26 possible values for $X$ that the young girl can obtain in this case, and those values are listed to the right.
Some values of $X$ are more common than others. For example, winning only 2 tickets in two spins is a somewhat common occurrence with probability 0.1225 as it means the player earns 1 ticket on the first spin and 1 ticket on the second spin. Similarly, winning 200 tickets in two spins is a somewhat rare occurrence with probability 0.0001 as it means the player earns 100 tickets on the first spin and 100 tickets on the second spin. A full list of the possible values of $X$ and the corresponding probabilities for almost every value of $X$ is shown at right.

a. Four probability values are deliberately hidden. Determine the 4 missing probability values in the distribution. (Hint: since all values of $X$ are listed, and since the probabilities that are shown add up to 0.66, the 4 hidden probabilities you are computing should add up to very specific value.)

b. Which value of $X$ is most common?

c. The young girl considers it a "good day" with the game if she wins more than 100 tickets based on 2 spins. What is the probability that she will have a "good day" based on that definition?

<table>
<thead>
<tr>
<th>$X$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1225</td>
</tr>
<tr>
<td>3</td>
<td>0.1400</td>
</tr>
<tr>
<td>4</td>
<td>0.0800</td>
</tr>
<tr>
<td>5</td>
<td>0.0400</td>
</tr>
<tr>
<td>6</td>
<td>0.0200</td>
</tr>
<tr>
<td>7</td>
<td>0.0100</td>
</tr>
<tr>
<td>8</td>
<td>0.0200</td>
</tr>
<tr>
<td>9</td>
<td>0.0100</td>
</tr>
<tr>
<td>10</td>
<td>0.0200</td>
</tr>
<tr>
<td>11</td>
<td>0.0100</td>
</tr>
<tr>
<td>12</td>
<td>0.0050</td>
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<tr>
<td>13</td>
<td>0.0025</td>
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<tr>
<td>14</td>
<td>0.0012</td>
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<tr>
<td>15</td>
<td>0.0006</td>
</tr>
<tr>
<td>16</td>
<td>0.0003</td>
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<tr>
<td>17</td>
<td>0.0001</td>
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<td>18</td>
<td>0.0000</td>
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<td>19</td>
<td>0.0000</td>
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<td>0.0000</td>
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<td>23</td>
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<td>25</td>
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<td>26</td>
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<td>28</td>
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<td>29</td>
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<td>47</td>
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<td>48</td>
<td>0.0000</td>
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<tr>
<td>49</td>
<td>0.0000</td>
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<td>50</td>
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<tr>
<td>51</td>
<td>0.0000</td>
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<td>52</td>
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<td>61</td>
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<td>67</td>
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<td>68</td>
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<td>69</td>
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<td>70</td>
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<td>71</td>
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<td>80</td>
<td>0.0000</td>
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<td>81</td>
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<td>82</td>
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<td>83</td>
<td>0.0000</td>
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<td>84</td>
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<td>85</td>
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<td>86</td>
<td>0.0000</td>
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<td>87</td>
<td>0.0000</td>
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<td>88</td>
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<td>89</td>
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<td>90</td>
<td>0.0000</td>
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<td>91</td>
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<td>99</td>
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<td>100</td>
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<td>101</td>
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<tr>
<td>102</td>
<td>0.0000</td>
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<td>103</td>
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<tr>
<td>104</td>
<td>0.0000</td>
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<tr>
<td>105</td>
<td>0.0000</td>
</tr>
<tr>
<td>106</td>
<td>0.0000</td>
</tr>
<tr>
<td>125</td>
<td>0.0000</td>
</tr>
<tr>
<td>200</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

c. Example: Solution (DOK 3)
The 10 cards below provide information on the ten students on the robotics team at a high school. The students are identified by an ID number: S1, S2, ..., S10. For each student, the following information is also given:

Gender

Grade level

Whether or not the student is currently enrolled in a science class

Whether or not the student is currently participating on a school sports team

Typical number of hours of sleep on a school night

It might be helpful to cut out these cards so that you can sort and rearrange them as you answer the following questions.

a. Suppose that one student on the robotics team will be selected at random to represent the team at an upcoming competition. This can be viewed as a chance experiment. Which one of the following is the sample space for this experiment?

\[ S = \{ \text{Student ID, gender, grade level, science class, sports team, hours of sleep} \} \]

\[ S = \{S1, S2, S3, S4, S5, S6, S7, S8, S9, S10\} \]

\[ S = \{S1, S2, S3, S4, S5, S6, S7, S8, S9, S10, \text{ male, female, 9, 10, 11, 12, yes, no, 6, 7, 8, 9}\} \]
b. What outcomes from the sample space are in the event that the selected student is taking a science class?

c. Consider the following three events:
   A = the selected student is female
   B = the selected students is on a sports team
   C = the selected student typically sleeps less than 8 hours on a school night

d. For each of these three events, list the outcomes that make up the event.

e. Which outcomes are in the following events?
   a. A or B
   b. A and C
   c. not C

f. Describe each of the events in question 4 in words.

g. Define two additional events in the context of this chance experiment. Use the letters D and E to represent these events.

h. For the two events you defined in question 6, describe the event D or E and the event D and E in words.

i. Is the set of outcomes in D or E and the set of outcomes in D and E the same for the two events you have defined? Explain why or why not.
<table>
<thead>
<tr>
<th>S1</th>
<th>S2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gender:</strong> Female</td>
<td><strong>Gender:</strong> Male</td>
</tr>
<tr>
<td><strong>Grade:</strong> 11</td>
<td><strong>Grade:</strong> 9</td>
</tr>
<tr>
<td><strong>Science course:</strong> yes</td>
<td><strong>Science course:</strong> no</td>
</tr>
<tr>
<td><strong>Sports team:</strong> no</td>
<td><strong>Sports team:</strong> no</td>
</tr>
<tr>
<td><strong>Sleep:</strong> 7 hours</td>
<td><strong>Sleep:</strong> 9 hours</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td><strong>Gender:</strong> Male</td>
<td><strong>Gender:</strong> Male</td>
</tr>
<tr>
<td><strong>Grade:</strong> 11</td>
<td><strong>Grade:</strong> 10</td>
</tr>
<tr>
<td><strong>Science course:</strong> no</td>
<td><strong>Science course:</strong> yes</td>
</tr>
<tr>
<td><strong>Sports team:</strong> yes</td>
<td><strong>Sports team:</strong> yes</td>
</tr>
<tr>
<td><strong>Sleep:</strong> 8 hours</td>
<td><strong>Sleep:</strong> 6 hours</td>
</tr>
<tr>
<td></td>
<td>S5</td>
</tr>
<tr>
<td>---</td>
<td>----------------------------------------</td>
</tr>
<tr>
<td></td>
<td>Gender: Female</td>
</tr>
<tr>
<td></td>
<td>Grade: 12</td>
</tr>
<tr>
<td></td>
<td>Science course: yes</td>
</tr>
<tr>
<td></td>
<td>Sports team: yes</td>
</tr>
<tr>
<td></td>
<td>Sleep: 7 hours</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>S7</td>
</tr>
<tr>
<td></td>
<td>Gender: Male</td>
</tr>
<tr>
<td></td>
<td>Grade: 11</td>
</tr>
<tr>
<td></td>
<td>Science course: yes</td>
</tr>
<tr>
<td></td>
<td>Sports team: no</td>
</tr>
<tr>
<td></td>
<td>Sleep: 7 hours</td>
</tr>
</tbody>
</table>
2. Understand that two events $A$ and $B$ are independent if the probability of $A$ and $B$ occurring together is the product of their probabilities, and use this characterization to determine if they are independent. (S-CP.A.2) (DOK 1)

a. Example: Solution (DOK 3)

One card is selected at random from the following set of 6 cards, each of which has a number and a black or white symbol:
\[ \{2\triangle, 4\blacksquare, 8\blacksquare, 8\heart, 5\blacksquare, 5\blacksquare\} \]

a. Let $B$ be the event that the selected card has a black symbol, and $F$ be the event that the selected card has a 5. Are the events $B$ and $F$ independent? Justify your answer with appropriate calculations.

b. Let $B$ be the event that the selected card has a black symbol, and $E$ be the event that the selected card has an 8. Are the events $B$ and $E$ independent? Justify your answer with appropriate calculations.

b. Example: Solution (DOK 3)
On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Some believe that the rescue procedures favored the wealthier first class passengers. Data on survival of passengers are summarized in the table below. We will use this data to investigate the validity of such claims. (Data source: http://www.encyclopedia-titanica.org/titanic-statistics.html)

<table>
<thead>
<tr>
<th></th>
<th>Survived</th>
<th>Did not survive</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>First class passengers</td>
<td>201</td>
<td>123</td>
<td>324</td>
</tr>
<tr>
<td>Second class passengers</td>
<td>118</td>
<td>166</td>
<td>284</td>
</tr>
<tr>
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<td>181</td>
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a. Are the events “passenger survived” and “passenger was in first class” independent events? Support your answer using appropriate probability calculations.

b. Are the events “passenger survived” and “passenger was in third class” independent events? Support your answer using appropriate probability calculations.

c. Did all passengers aboard the Titanic have the same probability of surviving? Support your answer using appropriate probability calculations.

c. Example: **Solution** (DOK 3)

a. Today there is a 55% chance of rain, a 20% chance of lightning, and a 15% chance of lightning and rain together. Are the two events “rain today” and “lightning today” independent events? Justify your answer.

b. Now suppose that today there is a 60% chance of rain, a 15% chance of lightning, and a 20% chance of lightning if it’s raining. What is the chance of both rain and lightning today?

c. Now suppose that today there is a 55% chance of rain, a 20% chance of lightning, and a 15% chance of lightning and rain. What is the chance that we will have rain or lightning today?

d. Now suppose that today there is a 50% chance of rain, a 60% chance of rain or lightning, and a 15% chance of rain and lightning. What is the chance that we will have lightning today?

d. Example: **Solution** (DOK 3)
In order to play a popular “spinning wheel” game at Fred’s Fun Factory Arcade, a player is required to pay a small, fixed amount of 25 cents each time he/she wants to make the wheel spin. When the wheel stops, the player is awarded tickets based on where the wheel stops -- and these tickets are then redeemable for prizes at a redemption center within the arcade.

This particular game has no skill component; each spin of the wheel is a random event, and the results from each spin of the wheel are independent of the results of previous spins.

The wheel awards tickets with the following probabilities:

<table>
<thead>
<tr>
<th>Tickets</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35%</td>
</tr>
<tr>
<td>2</td>
<td>20%</td>
</tr>
<tr>
<td>3</td>
<td>20%</td>
</tr>
<tr>
<td>5</td>
<td>10%</td>
</tr>
<tr>
<td>10</td>
<td>10%</td>
</tr>
<tr>
<td>25</td>
<td>4%</td>
</tr>
<tr>
<td>100</td>
<td>1%</td>
</tr>
</tbody>
</table>

A young girl is given 2 quarters so that she can play the game two times. Let \( X \) be the number of tickets she wins based on two spins. There are 26 possible values for \( X \) that the young girl can obtain in this case, and those values are listed to the right.

Some values of \( X \) are more common than others. For example, winning only 2 tickets in two spins is a somewhat common occurrence with probability 0.1225 as it means the player earns 1 ticket on the first spin and 1 ticket on the second spin. Similarly, winning 200 tickets in two spins is a somewhat rare occurrence with probability 0.0001 as it means the player earns 100 tickets on the first spin and 100 tickets on the second spin. A full list of the possible values of \( X \) and the corresponding probabilities for almost every value of \( X \) is shown at right.

a. Four probability values are deliberately hidden. Determine the 4 missing probability values in the distribution. (Hint: since all values of \( X \) are listed, and since the probabilities that are shown add up to 0.66, the 4 hidden probabilities you are computing should add up to very specific value.)

b. Which value of \( X \) is most common?

c. The young girl considers it a "good day" with the game if she wins more than 100 tickets based on 2 spins. What is the probability that she will have a "good day" based on that definition?
3. Understand the conditional probability of $A$ given $B$ as $P(A \text{ and } B)/P(B)$, and interpret independence of $A$ and $B$ as saying that the conditional probability of $A$ given $B$ is the same as the probability of $A$, and the conditional probability of $B$ given $A$ is the same as the probability of $B$. \(\text{(S-CP.A.3) (DOK 1,2)}\)

a. Example: Solution (DOK 3)

One card is selected at random from the following set of 6 cards, each of which has a number and a black or white symbol:
\[
\{2\triangle, 4\blacksquare, 8\blacksquare, 8\lozenge, 5\blacksquare, 5\blacksquare\}
\]

a. Let $B$ be the event that the selected card has a black symbol, and $F$ be the event that the selected card has a 5. Are the events $B$ and $F$ independent? Justify your answer with appropriate calculations.

b. Let $B$ be the event that the selected card has a black symbol, and $E$ be the event that the selected card has an 8. Are the events $B$ and $E$ independent? Justify your answer with appropriate calculations.

b. Example: Solution (DOK 3)
There are four red envelopes, four blue envelopes, and four $1 bills, which will be placed in four of the eight envelopes. Define the event $A$ as “you pick a lucky envelope (one that has a $1 bill in it)” and event $B$ as “you pick a blue envelope”.

a. Suppose one $1 bill is placed in a blue envelope, and the three remaining $1 bills are placed in three red envelopes.

i. If you choose one envelope at random, what is the probability that you pick a lucky envelope? How would you write this probability symbolically (using letters $A$ and/or $B$)?

ii. If you know that the envelope you picked is blue, what is the probability that you picked a lucky envelope? How would you write this probability symbolically (using letters $A$ and/or $B$)?

iii. Did knowing that the envelope is blue change the probability of getting a lucky envelope?

b. Now suppose we redistributed the four $1 bills between two blue and two red envelopes.

i. If you choose one envelope at random, what is the probability that you pick a lucky envelope? How would you write this probability symbolically (using letters $A$ and/or $B$)?

ii. If you know that the envelope you picked is blue, what is the probability that you picked a lucky envelope? How would you write this probability symbolically (using letters $A$ and/or $B$)?

iii. Did knowing that the envelope is blue change the probability of getting a lucky envelope?

c. Two events are independent if knowing that one event has occurred has no effect on the probability that the other has occurred.

i. Are the events $A$ and $B$ from part (a) independent events?

ii. Are the events $A$ and $B$ from part (b) independent events?

iii. Suppose two events $E$ and $F$ are independent. What does the definition of independence imply about the two probabilities $P(E)$ and $P(E|F)$?

c. Example: Solution (DOK 3)
On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Some believe that the rescue procedures favored the wealthier first class passengers. Data on survival of passengers are summarized in the table below. We will use this data to investigate the validity of such claims. (Data source: http://www.encyclopedia-titanica.org/titanic-statistics.html)

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a. Are the events “passenger survived” and “passenger was in first class” independent events? Support your answer using appropriate probability calculations.

b. Are the events “passenger survived” and “passenger was in third class” independent events? Support your answer using appropriate probability calculations.

c. Did all passengers aboard the Titanic have the same probability of surviving? Support your answer using appropriate probability calculations.

d. Example: Solution (DOK 3)

   a. Today there is a 55% chance of rain, a 20% chance of lightning, and a 15% chance of lightning and rain together. Are the two events “rain today” and “lightning today” independent events? Justify your answer.

   b. Now suppose that today there is a 60% chance of rain, a 15% chance of lightning, and a 20% chance of lightning if it’s raining. What is the chance of both rain and lightning today?

   c. Now suppose that today there is a 55% chance of rain, a 20% chance of lightning, and a 15% chance of lightning and rain. What is the chance that we will have rain or lightning today?

   d. Now suppose that today there is a 50% chance of rain, a 60% chance of rain or lightning, and a 15% chance of rain and lightning. What is the chance that we will have lightning today?

   e. Example: Solution (DOK 3)
Cecil has two six-sided dice, a red one and a white one.

a. If Cecil throws the two dice, what is the probability that the red die is a 1? What is the probability that the sum of the dice is 7?

b. Are the two events described part (a) independent? Explain.

c. What is the probability that the red die is a 2? What is the probability that the sum of the two dice is 10?

d. Are the two events described in part (c) independent? Explain.

4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results. (S-CP.A.4) (DOK 1,2)

a. Example: Solution (DOK 3)

On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Data on survival of passengers are summarized in the table below. (Data source: http://www.encyclopedia-titanica.org/titanic-statistics.html)

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</table>

a. Calculate the following probabilities. Round your answers to three decimal places.

i. If one of the passengers is randomly selected, what is the probability that this passenger was in first class?

ii. If one of the passengers is randomly selected, what is the probability that this passenger survived?

iii. If one of the passengers is randomly selected, what is the probability that this passenger was in first class and survived?

iv. If one of the passengers is randomly selected from the first class passengers, what is the probability that this passenger survived? (That is, what is the probability that the passenger survived, given that this passenger was in first class?)
v. If one of the passengers who survived is randomly selected, what is the probability that this passenger was in first class?

vi. If one of the passengers who survived is randomly selected, what is the probability that this passenger was in third class?

b. Why is the answer to part (a.iv) larger than the answer to part (a.iii)?

c. Why is the answer to part (a.v) larger than the answer to part (a.vi)?

d. What other questions can you ask and answer using information in the given table? List at least three.

b. **Example: Solution (DOK 3)**

On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of its 2,204 passengers and crew surviving. Some believe that the rescue procedures favored the wealthier first class passengers. Data on survival of passengers are summarized in the table below. We will use this data to investigate the validity of such claims. (Data source: [http://www.encyclopedia-titanica.org/titanic-statistics.html](http://www.encyclopedia-titanica.org/titanic-statistics.html))

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b. Are the events “passenger survived” and “passenger was in third class” independent events? Support your answer using appropriate probability calculations.

c. Did all passengers aboard the Titanic have the same probability of surviving? Support your answer using appropriate probability calculations.

c. **Example: Solution (DOK 3)**
On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Some believe that the rescue procedures favored the wealthier first class passengers. Other believe that the survival rates can be explained by the “women and children first” policy. Data on survival of passengers are summarized in the table below. Investigate what might and might not be concluded from the given data. (Data source: http://www.encyclopedia-titanica.org/titanic-statistics.html)

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<th></th>
<th>Survived</th>
<th>Did not survive</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Children in first class</td>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Women in first class</td>
<td>139</td>
<td>4</td>
<td>143</td>
</tr>
<tr>
<td>Men in first class</td>
<td>58</td>
<td>118</td>
<td>176</td>
</tr>
<tr>
<td>Children in second class</td>
<td>22</td>
<td>0</td>
<td>22</td>
</tr>
<tr>
<td>Women in second class</td>
<td>83</td>
<td>12</td>
<td>95</td>
</tr>
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<td>13</td>
<td>154</td>
<td>167</td>
</tr>
<tr>
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<td>30</td>
<td>50</td>
<td>80</td>
</tr>
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<td>91</td>
<td>88</td>
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</tr>
</tbody>
</table>

d. Example: Solution (DOK 3)
All of the upper-division students (juniors and seniors) at a high school were classified according to grade level and response to the question "How do you usually get to school?" The resulting data are summarized in the two-way table below.

<table>
<thead>
<tr>
<th></th>
<th>Car</th>
<th>Bus</th>
<th>Walk</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Juniors</td>
<td>96</td>
<td>122</td>
<td>56</td>
<td>274</td>
</tr>
<tr>
<td>Seniors</td>
<td>184</td>
<td>58</td>
<td>30</td>
<td>272</td>
</tr>
<tr>
<td>Totals</td>
<td>280</td>
<td>180</td>
<td>86</td>
<td>546</td>
</tr>
</tbody>
</table>

a. If an upper-division student at this school is selected at random, what is the probability that this student usually takes a bus to school?

i. \( \frac{58}{272} \)

ii. \( \frac{180}{546} \)

iii. \( \frac{122}{274} \)

iv. \( \frac{58}{122} \)

v. \( \frac{272}{546} \)

b. If a randomly selected upper-division student says he or she is a junior, what is the probability that he or she usually walks to school?

i. \( \frac{56}{546} \)

ii. \( \frac{86}{546} \)

iii. \( \frac{56}{274} \)

iv. \( \frac{86}{274} \)

v. \( \frac{274}{546} \)

e. Example: Solution (DOK 3)
A certain test for mononucleosis has a 99% chance of correctly diagnosing a patient with mononucleosis and a 5% chance of misdiagnosing a patient who does not have the infection. Suppose the test is given to a group where 1% of the people have mononucleosis. If a randomly selected patient’s test result is positive, what is the probability that she has mononucleosis? Explain.

f. Example: Solution (DOK 3)
   Each student in a random sample of seniors at a local high school participated in a survey. These students were asked to indicate their gender and their eye color. The following table summarizes the results of the survey.

<table>
<thead>
<tr>
<th>Gender</th>
<th>Brown</th>
<th>Blue</th>
<th>Green</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>50</td>
<td>40</td>
<td>20</td>
<td>110</td>
</tr>
<tr>
<td>Female</td>
<td>40</td>
<td>40</td>
<td>10</td>
<td>90</td>
</tr>
<tr>
<td>Total</td>
<td>90</td>
<td>80</td>
<td>30</td>
<td>200</td>
</tr>
</tbody>
</table>

a. Suppose that one of these seniors is randomly selected. What is the probability that the selected student is a male?

b. Suppose that one of these seniors is randomly selected. What is the probability that the selected student has blue eyes?

c. Suppose that one of these seniors is randomly selected. What is the probability that the selected student is a male and has blue eyes?
d. Suppose that one of these seniors is randomly selected. What is the probability that the selected student is a male or has blue eyes?

e. Suppose that one of these seniors is randomly selected. What is the probability that the selected student has blue eyes, given that the student is male?

f. Suppose that one of these seniors is randomly selected. What is the probability that the selected student is a male, given that the student has blue eyes?

5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer. (S-CP.A.5) (DOK 1,2,3)

   a. Example: Solution (DOK 3)

   On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Some believe that the rescue procedures favored the wealthier first class passengers. Data on survival of passengers are summarized in the table below. We will use this data to investigate the validity of such claims. (Data source: http://www.encyclopedia-titanica.org/titanic-statistics.html)

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   b. Are the events “passenger survived” and “passenger was in third class” independent events? Support your answer using appropriate probability calculations.

   c. Did all passengers aboard the Titanic have the same probability of surviving? Support your answer using appropriate probability calculations.

   b. Example: Solution (DOK 3)
On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Some believe that the rescue procedures favored the wealthier first class passengers. Other believe that the survival rates can be explained by the “women and children first” policy. Data on survival of passengers are summarized in the table below. Investigate what might and might not be concluded from the given data. (Data source: http://www.encyclopedia-titanica.org/titanic-statistics.html)

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c. Example: Solution (DOK 3)

On school days, Janelle sometimes eats breakfast and sometimes does not. After studying probability for a few days, janelle says, “The events ‘I eat breakfast’ and ‘I am late for school’ are independent.” Explain what this means in terms of the relationship between Janelle eating breakfast and her probability of being late for school in language that someone who hasn’t taken statistics would understand.

d. Example: Solution (DOK 3)
a. Today there is a 55% chance of rain, a 20% chance of lightning, and a 15% chance of lightning and rain together. Are the two events “rain today” and “lightning today” independent events? Justify your answer.

b. Now suppose that today there is a 60% chance of rain, a 15% chance of lightning, and a 20% chance of lightning if it’s raining. What is the chance of both rain and lightning today?

c. Now suppose that today there is a 55% chance of rain, a 20% chance of lightning, and a 15% chance of lightning and rain. What is the chance that we will have rain or lightning today?

d. Now suppose that today there is a 50% chance of rain, a 60% chance of rain or lightning, and a 15% chance of rain and lightning. What is the chance that we will have lightning today?

e. Example: Solution (DOK 3)
A seven-year-old boy has a favorite treat, Super Fruity Fruit Snax.

These "Fruit Snax" come in pouches of 10 snack pieces per pouch, and the pouches are generally sold by the box, with each box containing 4 pouches.

The snack pieces come in 5 different fruit flavors, and usually each pouch contains at least one piece from each of the 5 flavors. The website of the company that manufactures the product says that equal numbers of each of the 5 fruit flavors are produced and that pouches are filled in such a way that each piece added to a pouch is equally likely to be any one of the five flavors.

Of all the 5 fruit flavors, the seven-year-old boy likes mango the best. One day, he was very disappointed when he opened a pouch and there were no (zero) mango flavored pieces in the pouch. His mother (a statistician) assured him that this was no big deal and just happens by chance sometimes.

a. If the information on the company’s website is correct,

   i. What proportion of the population of snack pieces is mango flavored?

   ii. On average, how many mango flavored pieces should the boy expect in a pouch of 10 snack pieces?
iii. What is the chance that a pouch of 10 would have no mango flavored pieces? Was the mother's statement reasonable? Explain. (Hint: if none of the 10 independently selected pieces are mango, then all 10 pieces are "not mango.")

iv. The family then finds out that there were in fact no mango flavored pieces in any of the 4 pouches in the box they purchased. Again, if the information on the company's website is correct,
   i. What is the chance that an entire box of 4 pouches would have no mango flavored pieces? (Hint: How is this related to your answer to question (iii) in part (a)?)
   
   ii. Based on your answer and based on the fact that this event of an entire box with "no mangoes" happened to this family, would you be concerned about the company's claims, or would you say that such an event is not surprising given the company's claims? Explain.

Use the rules of probability to compute probabilities of compound events in a uniform probability model (S-CP.B)

6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model. (S-CP.B.6) (DOK 1,2)
   a. Example: Solution (DOK 3)
On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Data on survival of passengers are summarized in the table below. (Data source: http://www.encyclopedia-titanica.org/titanic-statistics.html)

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</tr>
<tr>
<td>Third class passengers</td>
<td>181</td>
<td>528</td>
<td>709</td>
</tr>
<tr>
<td>Total passengers</td>
<td>500</td>
<td>817</td>
<td>1317</td>
</tr>
</tbody>
</table>

a. Calculate the following probabilities. Round your answers to three decimal places.

i. If one of the passengers is randomly selected, what is the probability that this passenger was in first class?

ii. If one of the passengers is randomly selected, what is the probability that this passenger survived?

iii. If one of the passengers is randomly selected, what is the probability that this passenger was in first class and survived?

iv. If one of the passengers is randomly selected from the first class passengers, what is the probability that this passenger survived? (That is, what is the probability that the passenger survived, given that this passenger was in first class?)

v. If one of the passengers who survived is randomly selected, what is the probability that this passenger was in first class?

vi. If one of the passengers who survived is randomly selected, what is the probability that this passenger was in third class?

b. Why is the answer to part (a.iv) larger than the answer to part (a.iii)?

c. Why is the answer to part (a.v) larger than the answer to part (a.vi)?

d. What other questions can you ask and answer using information in the given table? List at least three.

b. Example: Solution (DOK 3)
On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Some believe that the rescue procedures favored the wealthier first class passengers. Data on survival of passengers are summarized in the table below. We will use this data to investigate the validity of such claims. (Data source: http://www.encyclopedia-titanica.org/titanic-statistics.html)

<table>
<thead>
<tr>
<th></th>
<th>Survived</th>
<th>Did not survive</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>First class passengers</td>
<td>201</td>
<td>123</td>
<td>324</td>
</tr>
<tr>
<td>Second class passengers</td>
<td>118</td>
<td>166</td>
<td>284</td>
</tr>
<tr>
<td>Third class passengers</td>
<td>181</td>
<td>528</td>
<td>709</td>
</tr>
<tr>
<td>Total passengers</td>
<td>500</td>
<td>817</td>
<td>1317</td>
</tr>
</tbody>
</table>

a. Are the events “passenger survived” and “passenger was in first class” independent events? Support your answer using appropriate probability calculations.

b. Are the events “passenger survived” and “passenger was in third class” independent events? Support your answer using appropriate probability calculations.

c. Did all passengers aboard the Titanic have the same probability of surviving? Support your answer using appropriate probability calculations.

c. Example: Solution (DOK 3)
On April 15, 1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Some believe that the rescue procedures favored the wealthier first class passengers. Others believe that the survival rates can be explained by the “women and children first” policy. Data on survival of passengers are summarized in the table below. Investigate what might and might not be concluded from the given data. (Data source: http://www.encyclopedia-titanica.org/titanic-statistics.html)

<table>
<thead>
<tr>
<th></th>
<th>Survived</th>
<th>Did not survive</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Children in first class</td>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Women in first class</td>
<td>139</td>
<td>4</td>
<td>143</td>
</tr>
<tr>
<td>Men in first class</td>
<td>58</td>
<td>118</td>
<td>176</td>
</tr>
<tr>
<td>Children in second class</td>
<td>22</td>
<td>0</td>
<td>22</td>
</tr>
<tr>
<td>Women in second class</td>
<td>83</td>
<td>12</td>
<td>95</td>
</tr>
<tr>
<td>Men in second class</td>
<td>13</td>
<td>154</td>
<td>167</td>
</tr>
<tr>
<td>Children in third class</td>
<td>30</td>
<td>50</td>
<td>80</td>
</tr>
<tr>
<td>Women in third class</td>
<td>91</td>
<td>88</td>
<td>179</td>
</tr>
<tr>
<td>Men in third class</td>
<td>60</td>
<td>390</td>
<td>450</td>
</tr>
<tr>
<td>Total passengers</td>
<td>500</td>
<td>817</td>
<td>1317</td>
</tr>
</tbody>
</table>

7. Apply the Addition Rule, $P(A\text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. (S-CP.B.7) (DOK 1,2)

   a. Example: Solution (DOK 3)

At Mom’s diner, everyone drinks coffee. Let $C$ = the event that a randomly-selected customer puts cream in their coffee. Let $S$ = the event that a randomly-selected customer puts sugar in their coffee. Suppose that after years of collecting data, Mom has estimated the following probabilities:

$$P(C) = 0.6$$
$$P(S) = 0.5$$
$$P(C \text{ or } S) = 0.7$$

Estimate $P(C \text{ and } S)$ and interpret this value in the context of the problem.

b. Example: Solution (DOK 3)
a. Today there is a 55% chance of rain, a 20% chance of lightning, and a 15% chance of lightning and rain together. Are the two events "rain today" and "lightning today" independent events? Justify your answer.

b. Now suppose that today there is a 60% chance of rain, a 15% chance of lightning, and a 20% chance of lightning if it's raining. What is the chance of both rain and lightning today?

c. Now suppose that today there is a 55% chance of rain, a 20% chance of lightning, and a 15% chance of lightning and rain. What is the chance that we will have rain or lightning today?

d. Now suppose that today there is a 50% chance of rain, a 60% chance of rain or lightning, and a 15% chance of rain and lightning. What is the chance that we will have lightning today?

c. Example: Solution
Consider the population consisting of the students at a high school who are members of the student council.

A diagram can be used to visualize this population. To construct such a diagram, a rectangle is drawn to represent the population, and each student on the student council is represented by a labeled point in the rectangle.

Here is the diagram for this population:

\[ \text{Diagram showing a rectangle with labeled points} \]

a. How many students are on this student council?

b. The given diagram also represents the sample space for the chance experiment of selecting a student at random from this population. For this chance experiment, what is the probability that student K is selected? Justify your answer.
c. In a diagram, an event can be represented by a shape (often a circle or oval) that encloses the outcomes that make up the event. Consider the following events:

\[ F = \text{the event that the selected student is a freshman} \]
\[ S = \text{the event that the selected student is a sophomore} \]
\[ J = \text{the event that the selected student is a junior} \]

Which of the following diagrams could be a representation of these three events? Explain your choice.

Diagram 1

![Diagram 1](image1)

Diagram 2

![Diagram 2](image2)

Use Diagram 2 above to answer questions d – f.

d. If a student is selected at random from the council, what is \( P(J) \), the probability that the selected student is a junior? Justify your answer.

e. Is \( P(F) = P(J) \)? Explain why or why not.

f. Use Diagram 2 to find the following probabilities:

\[ a. P(F) \]
\[ b. P(S) \]
\[ c. P(F \text{ or } S) \]

Now consider Diagram 3 shown below. This diagram contains the two events
\[ j = \text{event that the selected student is a junior} \]

\[ M = \text{event that the selected student is male} \]

Diagram 3

\[ g. \text{Shade the region where the two ovals overlap. What event is represented by this shaded region?} \]

\[ h. \text{Use Diagram 3 to find the following probabilities:} \]

\[ a. P(j) \]

\[ b. P(M) \]

\[ c. P(j \text{ or } M) \]

\[ d. P(j \text{ and } M) \]
i. Explain why \( P(F \text{ or } S) = P(F) + P(S) \) (see question 6) but \( P(F \text{ or } M) \neq P(F) + P(M) \).

j. Can you come up with a way to calculate \( P(F \text{ or } M) \) using the other three probabilities calculated in question h? Write this as a formula in the form

\[
P(F \text{ or } M) =
\]

k. Explain why your formula in question j makes sense in terms of Diagram 3.

l. In general, suppose \( A \) and \( B \) are two events. Write a formula for calculating \( P(A \text{ or } B) \) in each of the following two cases:

Case 1: \( A \) and \( B \) are mutually exclusive (\( A \) and \( B \) have no outcomes in common)
Case 2: \( A \) and \( B \) are not mutually exclusive

m. The formula for Case 2 in question l is called the General Addition Rule. This rule also works for calculating \( P(A \text{ or } B) \) when \( A \) and \( B \) are mutually exclusive. Explain why this is the case.

n. Write a sentence or two explaining how to calculate the probability of the union of two events.

8. (+) Apply the general Multiplication Rule in a uniform probability model, \( P(A \text{ and } B) = P(A)P(B | A) = P(B)P(A | B) \), and interpret the answer in terms of the model. (S-CP.B.8) (DOK 1,2)
   a. Example: Solution (DOK 3)

   A certain test for mononucleosis has a 99% chance of correctly diagnosing a patient with mononucleosis and a 5% chance of misdiagnosing a patient who does not have the infection. Suppose the test is given to a group where 1% of the people have mononucleosis. If a randomly selected patient’s test result is positive, what is the probability that she has mononucleosis? Explain.

9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. (S-CP.B.9) (DOK 1,2)
   a. Example: Solution (DOK 3)
Imagine Scott stood at zero on a life-sized number line. His friend flipped a coin 6 times. When the coin came up heads, he moved one unit to the right. When the coin came up tails, he moved one unit to the left. After each flip of the coin, Scott’s friend recorded his position on the number line. Let $f$ assign to the whole number $n$, when $1 \leq n \leq 6$, Scott’s position on the number line after the $n^{th}$ coin flip.

a. How many different outcomes are there for the sequence of 6 coin tosses?

b. Calculate the probability, before the coin flips have begun, that $f(6) = 0$, $f(6) = 1$, and $f(6) = 6$.

c. Make a bar graph showing the frequency of the different outcomes for this random walk.

d. Which number is Scott most likely to land on after the six coin flips? Why?

b. Example: Solution (DOK 3)
Imagine Scott stood at zero on a life-sized number line. His friend flipped a coin 100 times. When the coin came up heads, he moved one unit to the right. When the coin came up tails, he moved one unit to the left. After each flip of the coin, Scott’s friend recorded his position on the number line. Let $f$ assign to the whole number $n$, when $1 \leq n \leq 100$, Scott’s position on the number line after the $n^{th}$ coin flip.

a. Before the tosses begin, where is Scott most likely to be after ten coin tosses?

b. Before the coin tosses begin, what is the most likely value for $f(15)$? Explain.

c. Before the coin tosses begin, what is the most likely value for $f(100)$? Explain.

c. Example: Solution (DOK 2)
Alex, Mel, and Chelsea play a game that has 6 rounds. In each round there is a single winner, and the outcomes of the rounds are independent. For each round the probability that Alex wins is $\frac{1}{2}$, and Mel is twice as likely to win as Chelsea. What is the probability that Alex wins three rounds, Mel wins two rounds, and Chelsea wins one round?

d. Example: Solution (DOK 3)
In order to play a popular “spinning wheel” game at Fred’s Fun Factory Arcade, a player is required to pay a small, fixed amount of 25 cents each time he/she wants to make the wheel spin. When the wheel stops, the player is awarded tickets based on where the wheel stops -- and these tickets are then redeemable for prizes at a redemption center within the arcade.

This particular game has no skill component; each spin of the wheel is a random event, and the results from each spin of the wheel are independent of the results of previous spins.

The wheel awards tickets with the following probabilities:

<table>
<thead>
<tr>
<th>Tickets</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ticket</td>
<td>35%</td>
</tr>
<tr>
<td>2 tickets</td>
<td>20%</td>
</tr>
<tr>
<td>3 tickets</td>
<td>20%</td>
</tr>
<tr>
<td>5 tickets</td>
<td>10%</td>
</tr>
<tr>
<td>10 tickets</td>
<td>10%</td>
</tr>
<tr>
<td>25 tickets</td>
<td>4%</td>
</tr>
<tr>
<td>100 tickets</td>
<td>1%</td>
</tr>
</tbody>
</table>

A young girl is given 2 quarters so that she can play the game two times. Let $X$ be the number of tickets she wins based on two spins. There are 26 possible values for $X$ that the young girl can obtain in this case, and those values are listed to the right.
Some values of $X$ are more common than others. For example, winning only 2 tickets in two spins is a somewhat common occurrence with probability 0.1225 as it means the player earns 1 ticket on the first spin and 1 ticket on the second spin. Similarly, winning 200 tickets in two spins is a somewhat rare occurrence with probability 0.0001 as it means the player earns 100 tickets on the first spin and 100 tickets on the second spin. A full list of the possible values of $X$ and the corresponding probabilities for almost every value of $X$ is shown at right.

a. Four probability values are deliberately hidden. Determine the 4 missing probability values in the distribution. (Hint: since all values of $X$ are listed, and since the probabilities that are shown add up to 0.66, the 4 hidden probabilities you are computing should add up to very specific value.)

b. Which value of $X$ is most common?

c. The young girl considers it a "good day" with the game if she wins more than 100 tickets based on 2 spins. What is the probability that she will have a "good day" based on that definition?

<table>
<thead>
<tr>
<th>X</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1225</td>
</tr>
<tr>
<td>3</td>
<td>0.1400</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.0800</td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.0400</td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0700</td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.0400</td>
</tr>
<tr>
<td>13</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.0200</td>
</tr>
<tr>
<td>20</td>
<td>0.0100</td>
</tr>
<tr>
<td>26</td>
<td>0.0280</td>
</tr>
<tr>
<td>27</td>
<td>0.0160</td>
</tr>
<tr>
<td>28</td>
<td>0.0160</td>
</tr>
<tr>
<td>30</td>
<td>0.0080</td>
</tr>
<tr>
<td>35</td>
<td>0.0080</td>
</tr>
<tr>
<td>50</td>
<td>0.0016</td>
</tr>
<tr>
<td>101</td>
<td>0.0070</td>
</tr>
<tr>
<td>102</td>
<td>0.0040</td>
</tr>
<tr>
<td>103</td>
<td>0.0040</td>
</tr>
<tr>
<td>105</td>
<td>0.0020</td>
</tr>
<tr>
<td>110</td>
<td>0.0020</td>
</tr>
<tr>
<td>125</td>
<td>0.0008</td>
</tr>
<tr>
<td>200</td>
<td>0.0001</td>
</tr>
</tbody>
</table>
Calculate expected values and use them to solve problems (S-MD.A)

1. (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions. (S-MD.A.1) (DOK 1,2)

2. (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution. (S-MD.A.2) (DOK 1,2)
   a. Example: Solution (DOK 3)

   Bob’s Bagel Shop has estimated the following probabilities for the number of bagels a randomly-selected customer buys when he or she enters the shop.

<table>
<thead>
<tr>
<th>No. of Bagels</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.5</td>
</tr>
<tr>
<td>2</td>
<td>.1</td>
</tr>
<tr>
<td>3</td>
<td>.1</td>
</tr>
<tr>
<td>6</td>
<td>.1</td>
</tr>
<tr>
<td>12</td>
<td>.2</td>
</tr>
</tbody>
</table>

   Bob sells bagels for $0.80 each, or $9.00 for a dozen. If $X$ = the amount of money Bob collects from a randomly-selected customer, find and interpret the expected value of $X$.

   b. Example: Solution (DOK 3)
A famous arcade in a seaside resort town consists of many different games of skill and chance. In order to play a popular "spinning wheel" game at Fred's Fun Factory Arcade, a player is required to pay a small, fixed amount of 25 cents each time he/she wants to make the wheel spin. When the wheel stops, the player is awarded tickets based on where the wheel stops -- and these tickets are then redeemable for prizes at a redemption center within the arcade.

Note: this particular game has no skill component; each spin of the wheel is a random event, and the result from each spin of the wheel is independent of the results of previous spins.

The wheel awards tickets with the following probabilities:

<table>
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<tr>
<th>Tickets</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ticket</td>
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<td>20%</td>
</tr>
<tr>
<td>3 tickets</td>
<td>20%</td>
</tr>
<tr>
<td>5 tickets</td>
<td>10%</td>
</tr>
<tr>
<td>10 tickets</td>
<td>10%</td>
</tr>
<tr>
<td>25 tickets</td>
<td>4%</td>
</tr>
<tr>
<td>100 tickets</td>
<td>1%</td>
</tr>
</tbody>
</table>

(Note: A picture of a wheel fitting these parameters is included.)
a. If a player were to play this game many, many times, what is the expected (average) number of tickets that the player would win from each spin?

b. The arcade often provides quarters to its customers in $5.00 rolls. Every day over the summer, a young boy obtains one of these quarter rolls and uses all of the quarters for the spinning wheel game. In the long run, what is the average number of tickets that this boy can expect to win each day using this strategy?

c. One of the redemption center prizes that the young boy is playing for is a trendy item that costs 300 tickets. It is also available at a store down the street for $4.99. Without factoring in any enjoyment gained from playing the game or from visiting the arcade, from a strictly monetary point of view, would you advise the boy to try and obtain this item based on arcade ticket winnings or to just go and buy the item at the store? Explain.

d. The histogram below summarizes the results of 90 summer days of the boy playing the game using his "$5 roll per day" strategy. The first bar in the histogram represents those days where 40 to 59 tickets were won.

![Histogram](image)

i. For approximately what percentage of the 90 days did the boy earn fewer than 100 tickets in a day?

ii. For approximately what percentage of the 90 days did the boy earn 200 or more tickets in a day?

iii. For approximately what percentage of the 90 days did the boy earn 300 or more tickets in a day?
e. To maintain the spinning wheel game machine, the arcade manager adopts a strategy of emptying the money box (that’s where the quarters go after they are inserted in the machine) each time she refills the machine with a new roll of tickets. The ticket refill rolls contain 5000 tickets, and the machine is designed to hold $1300 in quarters in its money box. Assume that the machine was fully loaded with 5000 tickets and had an empty money box when it was first used. Using the manager’s maintenance strategy, is there any chance that the money box could become completely full with quarters or overflow with quarters? Explain.

**Arcade Wheel**

<table>
<thead>
<tr>
<th>Color</th>
<th>Tickets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orange</td>
<td>1</td>
</tr>
<tr>
<td>Green</td>
<td>2</td>
</tr>
<tr>
<td>Blue</td>
<td>3</td>
</tr>
<tr>
<td>Yellow</td>
<td>5</td>
</tr>
<tr>
<td>Purple</td>
<td>10</td>
</tr>
<tr>
<td>Black</td>
<td>25</td>
</tr>
<tr>
<td>Red</td>
<td>100</td>
</tr>
</tbody>
</table>

**Graphics Note:** The wheel diagram was developed in Microsoft Excel using its "Pie Chart" graph building feature. The intent is that each wedge represents 1% of the pie (3.6 degrees). There are 35 orange wedges (each representing a win of 1 ticket) to correspond to the 35% probability of obtaining 1 ticket in a spin, 20 green wedges (each representing a win of 2 tickets) to correspond to the 20% probability of obtaining 2 tickets in a spin, and so on.
c. Example: Solution (DOK 3)
A friend of yours, Phil, writes to you asking about a new scratch-off lottery game. It costs $10 to play this game. There are two outcomes for the game (win, lose) and the probability that a player wins a game is 60%. A win results in $15, for a net win of $5.

The probability distribution for $X =$ the amount of money a player wins (or loses) in a single game is as follows:

<table>
<thead>
<tr>
<th>$X$</th>
<th>Probability of $X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+$5$</td>
<td>.60</td>
</tr>
<tr>
<td>$-$10$</td>
<td>.40</td>
</tr>
</tbody>
</table>

a. Compute the expected value of $X$.

b. Your friend wants to know if he should play this game many, many times to make some extra money because the 60% chance of winning $5 sounds really good. Based on your calculation in part (a), complete the following message to your friend Phil that clearly recommends whether or not he should play this game many, many times and explains how the value you computed in part (a) led you to that conclusion.

*Phil:*

*Regarding your idea that you should play this new lottery game many, many times to make some extra money, I think...*
3. (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes. (S-MD.A.3.) (DOK 1,2,3)

4. (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households? (S-MD.A.4.) (DOK 1,2,3)

Use probability to evaluate outcomes of decisions (S-MD.B)

5. (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.
   a. Find the expected payoff for a game of chance. For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.
   b. Evaluate and compare strategies on the basis of expected values. For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident. (S-MD.B.5) (DOK 1,2,3)

Example: Solution (DOK 3)
A famous arcade in a seaside resort town consists of many different games of skill and chance. In order to play a popular “spinning wheel” game at Fred’s Fun Factory Arcade, a player is required to pay a small, fixed amount of 25 cents each time he/she wants to make the wheel spin. When the wheel stops, the player is awarded tickets based on where the wheel stops -- and these tickets are then redeemable for prizes at a redemption center within the arcade.

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<td>1%</td>
</tr>
</tbody>
</table>

(Note: A picture of a wheel fitting these parameters is included.)
a. If a player were to play this game many, many times, what is the expected (average) number of tickets that the player would win from each spin?

b. The arcade often provides quarters to its customers in $5.00 rolls. Every day over the summer, a young boy obtains one of these quarter rolls and uses all of the quarters for the spinning wheel game. In the long run, what is the average number of tickets that this boy can expect to win each day using this strategy?

c. One of the redemption center prizes that the young boy is playing for is a trendy item that costs 300 tickets. It is also available at a store down the street for $4.99. Without factoring in any enjoyment gained from playing the game or from visiting the arcade, from a strictly monetary point of view, would you advise the boy to try and obtain this item based on arcade ticket winnings or to just go and buy the item at the store? Explain.

d. The histogram below summarizes the results of 90 summer days of the boy playing the game using his "$5 roll per day" strategy. The first bar in the histogram represents those days where 40 to 59 tickets were won.

![](chart.png)

i. For approximately what percentage of the 90 days did the boy earn fewer than 100 tickets in a day?

ii. For approximately what percentage of the 90 days did the boy earn 200 or more tickets in a day?

iii. For approximately what percentage of the 90 days did the boy earn 300 or more tickets in a day?
e. To maintain the spinning wheel game machine, the arcade manager adopts a strategy of emptying the money box (that's where the quarters go after they are inserted in the machine) each time she refills the machine with a new roll of tickets. The ticket refill rolls contain 5000 tickets, and the machine is designed to hold $1300 in quarters in its money box. Assume that the machine was fully loaded with 5000 tickets and had an empty money box when it was first used. Using the manager's maintenance strategy, is there any chance that the money box could become completely full with quarters or overflow with quarters? Explain.

**Arcade Wheel**

<table>
<thead>
<tr>
<th>Color</th>
<th>Tickets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orange</td>
<td>1</td>
</tr>
<tr>
<td>Green</td>
<td>2</td>
</tr>
<tr>
<td>Blue</td>
<td>3</td>
</tr>
<tr>
<td>Yellow</td>
<td>5</td>
</tr>
<tr>
<td>Purple</td>
<td>10</td>
</tr>
<tr>
<td>Black</td>
<td>25</td>
</tr>
<tr>
<td>Red</td>
<td>100</td>
</tr>
</tbody>
</table>

Graphics Note: The wheel diagram was developed in Microsoft Excel using its "Pie Chart" graph building feature. The intent is that each wedge represents 1% of the pie (3.6 degrees). There are 35 orange wedges (each representing a win of 1 ticket) to correspond to the 35% probability of obtaining 1 ticket in a spin, 20 green wedges (each representing a win of 2 tickets) to correspond to the 20% probability of obtaining 2 tickets in a spin, and so on.

2. Example: **Solution** (DOK 3)
A friend of yours, Phil, writes to you asking about a new scratch-off lottery game. It costs $10 to play this game. There are two outcomes for the game (win, lose) and the probability that a player wins a game is 60%. A win results in $15, for a net win of $5.

The probability distribution for $X = \text{the amount of money a player wins (or loses) in a single game}$ is as follows:

<table>
<thead>
<tr>
<th>$X$</th>
<th>Probability of $X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+5$</td>
<td>.60</td>
</tr>
<tr>
<td>$-10$</td>
<td>.40</td>
</tr>
</tbody>
</table>

a. Compute the expected value of $X$.

b. Your friend wants to know if he should play this game many, many times to make some extra money because the 60% chance of winning $5 sounds really good. Based on your calculation in part (a), complete the following message to your friend Phil that clearly recommends whether or not he should play this game many, many times and explains how the value you computed in part (a) led you to that conclusion.

Phil:

Regarding your idea that you should play this new lottery game many, many times to make some extra money, I think...

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). (S-MD.B.6) (DOK 1,2)

7. Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). (S-MD.B.7) (DOK 2,3)
   a. Example: Solution (DOK 3)
A famous arcade in a seaside resort town consists of many different games of skill and chance. In order to play a popular “spinning wheel” game at Fred's Fun Factory Arcade, a player is required to pay a small, fixed amount of 25 cents each time he/she wants to make the wheel spin. When the wheel stops, the player is awarded tickets based on where the wheel stops -- and these tickets are then redeemable for prizes at a redemption center within the arcade.

Note: this particular game has no skill component; each spin of the wheel is a random event, and the result from each spin of the wheel is independent of the results of previous spins.

The wheel awards tickets with the following probabilities:

<table>
<thead>
<tr>
<th>Tickets</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ticket</td>
<td>35%</td>
</tr>
<tr>
<td>2 tickets</td>
<td>20%</td>
</tr>
<tr>
<td>3 tickets</td>
<td>20%</td>
</tr>
<tr>
<td>5 tickets</td>
<td>10%</td>
</tr>
<tr>
<td>10 tickets</td>
<td>10%</td>
</tr>
<tr>
<td>25 tickets</td>
<td>4%</td>
</tr>
<tr>
<td>100 tickets</td>
<td>1%</td>
</tr>
</tbody>
</table>

(Note: A picture of a wheel fitting these parameters is included.)

a. If a player were to play this game many, many times, what is the expected (average) number of tickets that the player would win from each spin?
b. The arcade often provides quarters to its customers in $5.00 rolls. Every day over the summer, a young boy obtains one of these quarter rolls and uses all of the quarters for the spinning wheel game. In the long run, what is the average number of tickets that this boy can expect to win each day using this strategy?

c. One of the redemption center prizes that the young boy is playing for is a trendy item that costs 300 tickets. It is also available at a store down the street for $4.99. Without factoring in any enjoyment gained from playing the game or from visiting the arcade, from a strictly monetary point of view, would you advise the boy to try and obtain this item based on arcade ticket winnings or to just go and buy the item at the store? Explain.

d. The histogram below summarizes the results of 90 summer days of the boy playing the game using his "$5 roll per day" strategy. The first bar in the histogram represents those days where 40 to 59 tickets were won.

e. To maintain the spinning wheel game machine, the arcade manager adopts a strategy of emptying the money box (that's where the quarters go after they are inserted in the machine) each time she refills the machine with a new roll of tickets. The ticket refill rolls contain 5000 tickets, and the machine is designed to hold $1300 in quarters in its money box. Assume that the machine was fully loaded with 5000 tickets and had an empty money box when it was first used. Using the manager's maintenance strategy, is there any chance that the money box could become completely full with quarters or overflow with quarters? Explain.

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Graphics Note: The wheel diagram was developed in Microsoft Excel using its "Pie Chart" graph building feature. The intent is that each wedge represents 1% of the pie (3.6 degrees). There are 35 orange wedges (each representing a win of 1 ticket) to correspond to the 35% probability of obtaining 1 ticket in a spin, 20 green wedges (each representing a win of 2 tickets) to correspond to the 20% probability of obtaining 2 tickets in a spin, and so on.

b. Example: Solution (DOK 3)
A seven-year-old boy has a favorite treat, Super Fruity Fruit Snax.

These "Fruit Snax" come in pouches of 10 snack pieces per pouch, and the pouches are generally sold by the box, with each box containing 4 pouches.

The snack pieces come in 5 different fruit flavors, and usually each pouch contains at least one piece from each of the 5 flavors. The website of the company that manufactures the product says that equal numbers of each of the 5 fruit flavors are produced and that pouches are filled in such a way that each piece added to a pouch is equally likely to be any one of the five flavors.

Of all the 5 fruit flavors, the seven-year-old boy likes mango the best. One day, he was very disappointed when he opened a pouch and there were no (zero) mango flavored pieces in the pouch. His mother (a statistician) assured him that this was no big deal and just happens by chance sometimes.

a. If the information on the company's website is correct,

   i. What proportion of the population of snack pieces is mango flavored?

   ii. On average, how many mango flavored pieces should the boy expect in a pouch of 10 snack pieces?

iii. What is the chance that a pouch of 10 would have no mango flavored pieces? Was the mother's statement reasonable? Explain. (Hint: if none of the 10 independently selected pieces are mango, then all 10 pieces are "not mango.")

iv. The family then finds out that there were in fact no mango flavored pieces in any of the 4 pouches in the box they purchased. Again, if the information on the company's website is correct,

   i. What is the chance that an entire box of 4 pouches would have no mango flavored pieces? (Hint: How is this related to your answer to question (iii) in part (a)?)

   ii. Based on your answer and based on the fact that this event of an entire box with "no mangoes" happened to this family, would you be concerned about the company's claims, or would you say that such an event is not surprising given the company's claims? Explain.
Note on courses and transitions

The high school portion of the Standards for Mathematical Content specifies the mathematics all students should study for college and career readiness. These standards do not mandate the sequence of high school courses. However, the organization of high school courses is a critical component to implementation of the standards. To that end, sample high school pathways for mathematics – in both a traditional course sequence (Algebra I, Geometry, and Algebra II) as well as an integrated course sequence (Mathematics 1, Mathematics 2, Mathematics 3) – will be made available shortly after the release of the final Common Core State Standards. It is expected that additional model pathways based on these standards will become available as well.

The standards themselves do not dictate curriculum, pedagogy, or delivery of content. In particular, states may handle the transition to high school in different ways. For example, many students in the U.S. today take Algebra I in the 8th grade, and in some states this is a requirement. The K-7 standards contain the prerequisites to prepare students for Algebra I by 8th grade, and the standards are designed to permit states to continue existing policies concerning Algebra I in 8th grade.

A second major transition is the transition from high school to post-secondary education for college and careers. The evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6-8. This body of material includes powerfully useful proficiencies such as applying ratio reasoning in real-world and mathematical problems, computing fluently with positive and negative fractions and decimals, and solving real-world and mathematical problems involving angle measure, area, surface area, and volume. Because important standards for college and career readiness are distributed across grades and courses, systems for evaluating college and career readiness should reach as far back in the standards as Grades 6-8. It is important to note as well that cut scores or other information generated by assessment systems for college and career readiness should be developed in collaboration with representatives from higher education and workforce development programs, and should be validated by subsequent performance of students in college and the workforce.

NOTE: The link listed below goes to the Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards document that is part of the Common Core State Standards for Mathematics. However, the document is not part of the Common Core Mathematics Standards that were adopted by Iowa’s State Board of Education. The pathways and courses addressed in the document are models not mandates.

http://www.corestandards.org/assets/CCSSI_Mathematics_Appendix_A.pdf
Glossary

Addition and subtraction within 5, 10, 20, 100, or 1000. Addition or subtraction of two whole numbers with whole number answers, and with sum or minuend in the range 0-5, 0-10, 0-20, or 0-100, respectively. Example: 8 + 2 = 10 is an addition within 10, 14 – 5 = 9 is a subtraction within 20, and 55 – 18 = 37 is a subtraction within 100.

Additive inverses. Two numbers whose sum is 0 are additive inverses of one another. Example: 3/4 and – 3/4 are additive inverses of one another because 3/4 + (– 3/4) = (– 3/4) + 3/4 = 0.

Associative property of addition. See Table 3 in this Glossary.

Associative property of multiplication. See Table 3 in this Glossary.

Bivariate data. Pairs of linked numerical observations. Example: a list of heights and weights for each player on a football team.

Box plot. A method of visually displaying a distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50% of the data.  

Commutative property. See Table 3 in this Glossary.

Complex fraction. A fraction A/B where A and/or B are fractions (B nonzero).

Computation algorithm. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also: computation strategy.

Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also: computation algorithm.

**Congruent.** Two plane or solid figures are congruent if one can be obtained from the other by rigid motion (a sequence of rotations, reflections, and translations).

**Counting on.** A strategy for finding the number of objects in a group without having to count every member of the group. For example, if a stack of books is known to have 8 books and 3 more books are added to the top, it is not necessary to count the stack all over again. One can find the total by counting on—pointing to the top book and saying “eight,” following this with “nine, ten, eleven. There are eleven books now.”

**Dot plot.** See: line plot.

**Dilation.** A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

**Expanded form.** A multi-digit number is expressed in expanded form when it is written as a sum of single-digit multiples of powers of ten. For example, 643 = 600 + 40 + 3.

**Expected value.** For a random variable, the weighted average of its possible values, with weights given by their respective probabilities.
**First quartile.** For a data set with median $M$, the first quartile is the median of the data values less than $M$. Example: For the data set \(\{1, 3, 6, 7, 10, 12, 14, 15, 22, 120\}\), the first quartile is 6. *See also:* median, third quartile, interquartile range.\(^2\)

**Fraction.** A number expressible in the form $a/b$ where $a$ is a whole number and $b$ is a positive whole number. (The word *fraction* in these standards always refers to a non-negative number.) *See also:* rational number.

**Identity property of 0.** See Table 3 in this Glossary.

**Independently combined probability models.** Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

**Integer.** A number expressible in the form $a$ or $-a$ for some whole number $a$.

**Interquartile Range.** A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. Example: For the data set \(\{1, 3, 6, 7, 10, 12, 14, 15, 22, 120\}\), the interquartile range is $15 - 6 = 9$. *See also:* first quartile, third quartile.

**Line plot.** A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. Also known as a dot plot.\(^3\)

**Mathematics of Information Processing and the Internet (IA).** The Internet is everywhere in modern life. To be informed consumers and citizens in the information-dense modern world permeated by the Internet, students should have a basic mathematical understanding of some of the issues of information processing on the Internet. For example, when making an online purchase, mathematics is used to help you find what you want, encrypt your credit card number so that you can safely buy it, send your order accurately to the vendor, and, if your order is immediately downloaded, as when purchasing software, music, or video, ensure that your download occurs quickly and error-free. Essential topics related to these aspects of information processing are basic set theory, logic, and modular arithmetic. These topics are not only fundamental to information processing on the Internet, but they are also important mathematical topics in their own right with applications in many other areas.

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\(^3\) Adapted from Wisconsin Department of Public Instruction, *op. cit.*
Mathematics of Voting (IA). The instant-runoff voting (IRV), the Borda method (assigning points for preferences), and the Condorcet method (in which each pair of candidates is run off head to head) are all forms of preferential voting (rank according to your preferences, rather than just voting for your single favorite candidate).

Mean. A measure of center in a set of numerical data, computed by adding the values in a list and then dividing by the number of values in the list. Example: For the data set \{1, 3, 6, 7, 10, 12, 14, 15, 22, 120\}, the mean is 21.

Mean absolute deviation. A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. Example: For the data set \{2, 3, 6, 7, 10, 12, 14, 15, 22, 120\}, the mean absolute deviation is 20.

Median. A measure of center in a set of numerical data. The median of a list of values is the value appearing at the center of a sorted version of the list—or the mean of the two central values, if the list contains an even number of values. Example: For the data set \{2, 3, 6, 7, 10, 12, 14, 15, 22, 90\}, the median is 11.

Midline. In the graph of a trigonometric function, the horizontal line halfway between its maximum and minimum values.

Multiplication and division within 100. Multiplication or division of two whole numbers with whole number answers, and with product or dividend in the range 0-100. Example: 72 ÷ 8 = 9.

Multiplicative inverses. Two numbers whose product is 1 are multiplicative inverses of one another. Example: \(\frac{3}{4}\) and \(\frac{4}{3}\) are multiplicative inverses of one another because \(\frac{3}{4} \times \frac{4}{3} = \frac{4}{3} \times \frac{3}{4} = 1\).

Number line diagram. A diagram of the number line used to represent numbers and support reasoning about them. In a number line diagram for measurement quantities, the interval from 0 to 1 on the diagram represents the unit of measure for the quantity.

Percent rate of change. A rate of change expressed as a percent. Example: if a population grows from 50 to 55 in a year, it grows by \(\frac{5}{50} = 10\%\) per year.

Probability distribution. The set of possible values of a random variable with a probability assigned to each.

---

4 To be more precise, this defines the arithmetic mean.
Properties of operations. See Table 3 in this Glossary.

Properties of equality. See Table 4 in this Glossary.

Properties of inequality. See Table 5 in this Glossary.

Properties of operations. See Table 3 in this Glossary.

Probability. A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, or testing for a medical condition).

Probability model. A probability model is used to assign probabilities to outcomes of a chance process by examining the nature of the process. The set of all outcomes is called the sample space, and their probabilities sum to 1. See also: uniform probability model.

Random variable. An assignment of a numerical value to each outcome in a sample space.

Rational expression. A quotient of two polynomials with a non-zero denominator.

Rational number. A number expressible in the form \( \frac{a}{b} \) or \( -\frac{a}{b} \) for some fraction \( \frac{a}{b} \). The rational numbers include the integers.

Rectilinear figure. A polygon all angles of which are right angles.

Rigid motion. A transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations. Rigid motions are here assumed to preserve distances and angle measures.

Repeating decimal. The decimal form of a rational number. See also: terminating decimal.

Sample space. In a probability model for a random process, a list of the individual outcomes that are to be considered.
**Scatter plot.** A graph in the coordinate plane representing a set of bivariate data. For example, the heights and weights of a group of people could be displayed on a scatter plot.\(^5\)

**Similarity transformation.** A rigid motion followed by a dilation.

**Tape diagram.** A drawing that looks like a segment of tape, used to illustrate number relationships. Also known as a strip diagram, bar model, fraction strip, or length model.

**Terminating decimal.** A decimal is called terminating if its repeating digit is 0.

**Third quartile.** For a data set with median \(M\), the third quartile is the median of the data values greater than \(M\). Example: For the data set \(\{2, 3, 6, 7, 10, 12, 14, 15, 22, 120\}\), the third quartile is 15. See also: median, first quartile, interquartile range.

**Transitivity principle for indirect measurement.** If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other quantities as well.

**Uniform probability model.** A probability model which assigns equal probability to all outcomes. See also: probability model.

**Vector.** A quantity with magnitude and direction in the plane or in space, defined by an ordered pair or triple of real numbers.

**Vertex-Edge Graphs (IA).** Vertex-edge graphs are diagrams consisting of vertices (points) and edges (line segments or arcs) connecting some of the vertices. Vertex-edge graphs are also sometimes called networks, discrete graphs, or finite graphs. A vertex-edge graph shows relationships and connections among objects, such as in a road network, a telecommunications network, or a family tree. Within the context of school geometry, which is fundamentally the study of shape, vertex-edge graphs represent, in a sense, the situation of no shape. That is, vertex-edge graphs are geometric models consisting of vertices and edges in which shape is not essential, only the connections among vertices are essential. These graphs are widely used in business and industry to solve problems about networks, paths, and relationships among a finite number of objects – such as, analyzing a computer

\(^5\) Adapted from Wisconsin Department of Public Instruction, *op. cit.*
network; optimizing the route used for snowplowing, collecting garbage, or visiting business clients; scheduling committee meetings to avoid conflicts; or planning a large construction project to finish on time.

**Visual fraction model.** A tape diagram, number line diagram, or area model.

**Whole numbers.** The numbers 0, 1, 2, 3, ....
Table 1. Common addition and subtraction situations.\textsuperscript{6}

<table>
<thead>
<tr>
<th>Table 1. Common addition and subtraction situations.\textsuperscript{6}</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Result Unknown</strong></td>
</tr>
<tr>
<td><strong>Add to</strong></td>
</tr>
<tr>
<td>Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Take from</strong></td>
</tr>
<tr>
<td>Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Put Together/ Take Apart\textsuperscript{2}</strong></td>
</tr>
<tr>
<td>Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Total Unknown</strong></td>
</tr>
<tr>
<td>(&quot;How many more?&quot; version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? 2 + ? = 5, 5 – 2 = ?</td>
</tr>
<tr>
<td>(&quot;How many fewer?&quot; version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Difference Unknown</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>1\textsuperscript{These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as.}</td>
</tr>
</tbody>
</table>

---

\textsuperscript{6}Adapted from Box 2-4 of Mathematics Learning in Early Childhood, National Research Council (2009, pp. 32,33).
Table 2. Common multiplication and division situations.\(^7\)

<table>
<thead>
<tr>
<th>Unknown Product</th>
<th>Group Size Unknown (&quot;How many in each group?&quot; Division)</th>
<th>Number of Groups Unknown (&quot;How many groups?&quot; Division)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 \times 6 = ?</td>
<td>3 \times ? = 18, and 18 ÷ 3 = ?</td>
<td>? \times 6 = 18, and 18 ÷ 6 = ?</td>
</tr>
<tr>
<td>Equal Groups</td>
<td>Measurement example. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?</td>
<td>Measurement example. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?</td>
</tr>
<tr>
<td>Arrays,(^4) Area(^5)</td>
<td>If 18 apples are arranged into 3 equal rows, how many apples will be in each row?</td>
<td>Measurement example. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?</td>
</tr>
<tr>
<td>Compare</td>
<td>Measurement example. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?</td>
<td>Area example. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?</td>
</tr>
<tr>
<td>General</td>
<td>a \times b = ?</td>
<td>a \times ? = p, and p ÷ a = ?</td>
</tr>
</tbody>
</table>

\(^4\)The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

\(^5\)Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

\(^7\)The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.
Table 3. The properties of operations. Here a, b and c stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

<table>
<thead>
<tr>
<th>Property of Operations</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associative property of addition</td>
<td>((a + b) + c = a + (b + c))</td>
</tr>
<tr>
<td>Commutative property of addition</td>
<td>(a + b = b + a)</td>
</tr>
<tr>
<td>Additive identity property of 0</td>
<td>(a + 0 = 0 + a = a)</td>
</tr>
<tr>
<td>Existence of additive inverses</td>
<td>For every (a) there exists (-a) so that (a + (-a) = (-a) + a = 0).</td>
</tr>
<tr>
<td>Associative property of multiplication</td>
<td>((a \times b) \times c = a \times (b \times c))</td>
</tr>
<tr>
<td>Commutative property of multiplication</td>
<td>(a \times b = b \times a)</td>
</tr>
<tr>
<td>Multiplicative identity property of 1</td>
<td>(a \times 1 = 1 \times a = a)</td>
</tr>
<tr>
<td>Existence of multiplicative inverses</td>
<td>For every (a \neq 0) there exists (1/\alpha) so that (a \times 1/\alpha = 1/\alpha \times a = 1).</td>
</tr>
<tr>
<td>Distributive property of multiplication over addition</td>
<td>(a \times (b + c) = a \times b + a \times c)</td>
</tr>
</tbody>
</table>

Table 4. The properties of equality. Here a, b and c stand for arbitrary numbers in the rational, real, or complex number systems.

<table>
<thead>
<tr>
<th>Property of Equality</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflexive property of equality</td>
<td>(a = a)</td>
</tr>
<tr>
<td>Symmetric property of equality</td>
<td>If (a = b), then (b = a).</td>
</tr>
<tr>
<td>Transitive property of equality</td>
<td>If (a = b) and (b = c), then (a = c).</td>
</tr>
<tr>
<td>Addition property of equality</td>
<td>If (a = b), then (a + c = b + c).</td>
</tr>
<tr>
<td>Subtraction property of equality</td>
<td>If (a = b), then (a - c = b - c).</td>
</tr>
<tr>
<td>Multiplication property of equality</td>
<td>If (a = b), then (a \times c = b \times c).</td>
</tr>
<tr>
<td>Division property of equality</td>
<td>If (a = b) and (c \neq 0), then (a \div c = b \div c).</td>
</tr>
<tr>
<td>Substitution property of equality</td>
<td>If (a = b), then (b) may be substituted for (a) in any expression containing (a).</td>
</tr>
</tbody>
</table>

Table 5. The properties of inequality. Here a, b and c stand for arbitrary numbers in the rational or real number systems.

<table>
<thead>
<tr>
<th>Property of Inequality</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exactly one of the following is true:</td>
<td>(a &lt; b, a = b, a &gt; b).</td>
</tr>
<tr>
<td>If (a &gt; b) and (b &gt; c) then (a &gt; c).</td>
<td></td>
</tr>
<tr>
<td>If (a &gt; b), then (b &lt; a).</td>
<td></td>
</tr>
<tr>
<td>If (a &gt; b), then (-a &lt; -b).</td>
<td></td>
</tr>
<tr>
<td>If (a &gt; b), then (a \pm c &gt; b \pm c).</td>
<td></td>
</tr>
<tr>
<td>If (a &gt; b) and (c &gt; 0), then (a \times c &gt; b \times c).</td>
<td></td>
</tr>
<tr>
<td>If (a &gt; b) and (c &lt; 0), then (a \times c &lt; b \times c).</td>
<td></td>
</tr>
<tr>
<td>If (a &gt; b) and (c &gt; 0), then (a \div c &gt; b \div c).</td>
<td></td>
</tr>
<tr>
<td>If (a &gt; b) and (c &lt; 0), then (a \div c &lt; b \div c).</td>
<td></td>
</tr>
</tbody>
</table>
Sample of Works Consulted

Existing state standards documents.

Research summaries and briefs provided to the Working Group by researchers.


Mathematics documents from: Alberta, Canada; Belgium; China; Chinese Taipei; Denmark; England; Finland; Hong Kong; India; Ireland; Japan; Korea; New Zealand; Singapore; Victoria (British Columbia).


Howe, R., “From Arithmetic to Algebra.”


Wu, H., “Crisis at the Core: Preparing All Students for College and Work,” ACT.


ACT College Readiness Benchmarks™

ACT College Readiness Standards™

ACT National Curriculum Survey™


Achieve, Inc., Virginia

ACT Job Skill Comparison Charts.


Hawai‘i Career Ready Study: access to living wage careers from high school, 2007.


ACT WorkKeys Occupational Profiles™.

Program for International Student Assessment (PISA), 2006.

Trends in International Mathematics and Science Study (TIMSS), 2007.


Individuals with Disabilities Education Act (IDEA), 34 CFR §300.34 (a). (2004).


The Role and Importance of Cognitive Complexity

The Iowa Core Standards for Mathematics are intended to play a central role in defining what teachers teach. That is, teachers are to align their instruction to the Standards. The Standards not only define the topical, procedural, and conceptual knowledge students are to learn, they also define the type of cognitive processes in which students are to engage. This is known as cognitive demand or cognitive complexity. The practical implication of cognitive complexity is that the Standards require teachers to provide students with instructional experiences that not only address the topical and conceptual knowledge of the standards, but the type of thinking called for by the standards as well. Compelling evidence suggests that when teachers align their instruction to an assessment, students perform better on that assessment. However, the impact of alignment is only detectable when both topical/conceptual knowledge and cognitive complexity are taken into consideration. (Gamoran, Porter, Smithson, & White, 1997)

The Iowa Core Standards for Mathematics have been coded for cognitive complexity using Webb’s Depth of Knowledge (DOK) approach (Webb, 2005). The DOK called for in each standard reflects the complexity of the standard, not its difficulty. The topical/conceptual knowledge detailed in a standard will be more or less difficult for each student, but requires a consistent level of complexity across students. The DOK of a standard describes the type of work students are most commonly required to perform to demonstrate their attainment of the standard. Webb’s DOK has four levels: DOK 1 = Recall, DOK 2 = Skills and Concepts, DOK 3 = Strategic Thinking, and DOK 4 = Extended Thinking. Detailed, verbatim descriptions of each level are provided next (Webb, 2002). These descriptions are intended to provide examples of the type of work students are expected to engage in for each standard.

Mathematics DOK Levels

Level 1 (Recall) includes the recall of information such as a fact, definition, term, or a simple procedure, as well as performing a simple algorithm or applying a formula. That is, in mathematics, a one-step, well defined, and straight algorithmic procedure should be included at this lowest level. Other key words that signify Level 1 include “identify,” “recall,” “recognize,” “use,” and “measure.” Verbs such as “describe” and “explain” could be classified at different levels, depending on what is to be described and explained.

Level 2 (Skill/Concept) includes the engagement of some mental processing beyond an habitual response. A Level 2 assessment item requires students to make some decisions as to how to approach the problem or activity, whereas Level 1 requires students to demonstrate a rote response, perform a well-known algorithm, follow a set procedure (like a recipe), or perform a clearly defined series of steps. Keywords that generally distinguish a Level 2 item include “classify,” “organize,” “estimate,” “make observations,” “collect and display data,” and “compare data.” These actions imply more than one step. For example, to compare data requires first identifying characteristics of objects or phenomena and then grouping or ordering the objects. Some action verbs, such as “explain,” “describe,” or “interpret,” could be classified at different levels depending on the object of the action. For example, interpreting information from a simple graph, or reading information from the graph, also are at Level 2. Interpreting information from a complex graph that requires some decisions on what features of the graph need to be considered and how information from the graph can be aggregated is at Level 3. Level 2 activities are not limited only to number skills, but may involve visualization skills and probability skills. Other Level 2 activities include noticing or describing non-trivial patterns, explaining the purpose and use of experimental procedures;
carrying out experimental procedures; making observations and collecting data; classifying, organizing, and comparing data; and organizing and displaying data in tables, graphs, and charts.

**Level 3 (Strategic Thinking)** requires reasoning, planning, using evidence, and a higher level of thinking than the previous two levels. In most instances, requiring students to explain their thinking is at Level 3. Activities that require students to make conjectures are also at this level. The cognitive demands at Level 3 are complex and abstract. The complexity does not result from the fact that there are multiple answers, a possibility for both Levels 1 and 2, but because the task requires more demanding reasoning. An activity, however, that has more than one possible answer and requires students to justify the response they give would most likely be at Level 3.

Other Level 3 activities include drawing conclusions from observations; citing evidence and developing a logical argument for concepts; explaining phenomena in terms of concepts; and deciding which concepts to apply in order to solve a complex problem.

**Level 4 (Extended Thinking)** requires complex reasoning, planning, developing, and thinking, most likely over an extended period of time. The extended time period is not a distinguishing factor if the required work is only repetitive and does not require applying significant conceptual understanding and higher-order thinking. For example, if a student has to take the water temperature from a river each day for a month and then construct a graph, this would be classified as a Level 2. However, if the student is to conduct a river study that requires taking into consideration a number of variables, this would be a Level 4. At Level 4, the cognitive demands of the task should be high and the work should be very complex. Students should be required to make several connections—relate ideas within the content area or among content areas—and have to select one approach among many alternatives on how the situation should be solved, in order to be at this highest level. Level 4 activities include designing and conducting experiments and projects; developing and proving conjectures, making connections between a finding and related concepts and phenomena; combining and synthesizing ideas into new concepts; and critiquing experimental designs.